Answer on Question #44531 – Math – Algebra

Let m, n, x, y, z be positive real numbers, with $x + y + z = 1$. Prove that

$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)}$ is greater than or equal to $\frac{1}{3(m+n)^2}$.

[Hint: Apply the AM ≥ GM inequality to each term in the LHS, and then apply the CS inequality.]

Solution

$$\frac{(a+b)^2}{4} = \left(\frac{a+b}{2}\right)^2 \geq \left(\frac{2\sqrt{ab}}{2}\right)^2 = ab,$$

by AM-GM inequality.

Hence

$$\frac{(m+n)^2(x+y)^2}{4} = \left(\frac{(mx+ny)+(my+nx)}{2}\right)^2 \geq (mx+ny)(my+nx),$$

$$\frac{(m+n)^2(y+z)^2}{4} = \left(\frac{(my+nz)+(mz+ny)}{2}\right)^2 \geq (my+nz)(mz+ny),$$

$$\frac{(m+n)^2(x+z)^2}{4} = \left(\frac{(mx+nz)+(mz+nx)}{2}\right)^2 \geq (mx+nz)(mz+nx).$$

Therefore

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)}$$
$$\geq \frac{4x^4}{(m+n)^2(x+y)^2} + \frac{4y^4}{(m+n)^2(y+z)^2} + \frac{4z^4}{(m+n)^2(x+z)^2}$$
$$= \frac{4}{(m+n)^2}\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2}\right).$$

$$\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2}\right)(1^2+1^2+1^2) \geq \left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)}\right)^2$$

by Cauchy-Shwartz inequality, so

$$\frac{4}{(m+n)^2}\left(\frac{x^4}{(x+y)^2} + \frac{y^4}{(y+z)^2} + \frac{z^4}{(x+z)^2}\right) \geq \frac{4}{3(m+n)^2}\left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)}\right)^2$$

and

$$\frac{x^4}{(mx+ny)(my+nx)} + \frac{y^4}{(my+nz)(mz+ny)} + \frac{z^4}{(mz+nx)(mx+nz)}$$
$$\geq \frac{4}{3(m+n)^2}\left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)}\right)^2.$$

$$\left(\frac{x^2}{(x+y)} + \frac{y^2}{(y+z)} + \frac{z^2}{(x+z)}\right)\big((x+y)+(y+z)+(x+z)\big) \geq (x+y+z)^2$$

by Cauchy-Shwartz inequality, so

$$\frac{4}{3(m+n)^2}\left(\frac{x^2}{(x+y)}+\frac{y^2}{(y+z)}+\frac{z^2}{(x+z)}\right)^2 \geq \frac{4}{3(m+n)^2}\frac{(x\ +\ y\ +\ z\ )^4}{\left((x+y)+(y+z)+(x+z)\right)^2}$$

$$=\frac{4}{3(m+n)^2}\frac{(1\ )^4}{(2)^2}=\frac{1}{3(m+n)^2}$$

as $x\ +\ y\ +\ z\ =\ 1$.

Hence

$$\frac{x^4}{(mx+ny)(my+nx)}+\frac{y^4}{(my+nz)(mz+ny)}+\frac{z^4}{(mz+nx)(mx+nz)} \geq \frac{1}{3(m+n)^2}.$$