Answer on Question #44484 – Math – Linear Algebra

Question. Use the properties of determinants to evaluate the following determinant:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}.$$

Solution. Multiply the second column by -1 and add it to the third column, the result is written to the third column:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (b+c)^2 & a^2 & 0 \\ b^2 & (c+a)^2 & (a+b+c)(\mathrm{b}-a-c) \\ c^2 & c^2 & (a+b+c)(a+b-c) \end{vmatrix}.$$

Further multiply the first column by -1 and add it to the second column, the result is written to the second column :

$$\begin{vmatrix} (b+c)^2 & a^2 & 0 \\ b^2 & (c+a)^2 & (a+b+c)(b-a-c) \\ c^2 & c^2 & (a+b+c)(a+b-c) \end{vmatrix} =$$

$$= \begin{vmatrix} (b+c)^2 & (\mathrm{a}+b+c)(\mathrm{a}-b-c) & 0 \\ b^2 & (a+b+c)(a+c-b) & (a+b+c)(b-a-c) \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

. Take the common factor $a+b+c$ from the second column and from the third column. We shall have

$$\begin{vmatrix} (b+c)^2 & (\mathrm{a}+b+c)(\mathrm{a}-b-c) & 0 \\ b^2 & (a+b+c)(a+c-b) & (\mathrm{a}+b+c)(b-a-c) \\ c^2 & 0 & (\mathrm{a}+b+c)(a+b-c) \end{vmatrix} =$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & \mathrm{a}-b-c & 0 \\ b^2 & a+c-b & b-a-c \\ c^2 & 0 & a+b-c \end{vmatrix}.$$

In the last determinant add the third column to the second column:

$$(a+b+c)^2 \begin{vmatrix} (b+c)^2 & \mathrm{a}-b-c & 0 \\ b^2 & a+c-b & b-a-c \\ c^2 & 0 & a+b-c \end{vmatrix} =$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & \mathrm{a}-b-c & 0 \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & a+b-c \end{vmatrix}$$

. Further add the second row to the third row:

$$(a+b+c)^2 \begin{vmatrix} (b+c)^2 & \mathrm{a}-b-c & 0 \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & a+b-c \end{vmatrix} =$$

$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & 0 & b-a-c \\ b^2+c^2 & a+b-c & 2b-2c \end{vmatrix}$. Multiply the third row by -1 and add it to the first row. Further take the common factor 2 from the first row:

$$(a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & 0 & b-a-c \\ b^2+c^2 & a+b-c & 2b-2c \end{vmatrix} =$$

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2b & 2c-2b \\ b^2 & 0 & b-a-c \\ b^2+c^2 & a+b-c & 2b-2c \end{vmatrix} =$$

$= 2(a+b+c)^2 \begin{vmatrix} bc & -b & c-b \\ b^2 & 0 & b-a-c \\ b^2+c^2 & a+b-c & 2b-2c \end{vmatrix}$. Multiply the second row by -1 and add it to the third row:

$$2(a+b+c)^2 \begin{vmatrix} bc & -b & c-b \\ b^2 & 0 & b-a-c \\ b^2+c^2 & a+b-c & 2b-2c \end{vmatrix} =$$

$= 2(a+b+c)^2 \begin{vmatrix} bc & -b & c-b \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & a+b-c \end{vmatrix}$. Multiply the second column by -1 and add it to the third column:

$$2(a+b+c)^2 \begin{vmatrix} bc & -b & c-b \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & a+b-c \end{vmatrix} =$$

$= 2(a+b+c)^2 \begin{vmatrix} bc & -b & c \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & 0 \end{vmatrix}$. We shall evaluate the last determinant using the rule of triangle:

$$2(a+b+c)^2 \begin{vmatrix} bc & -b & c \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & 0 \end{vmatrix} =$$

$= 2(a+b+c)^2[-bc^2(b-a-c) + b^2c(a+b-c) - bc(a+b-c)(b-a-c)] =$

$= 2bc(a+b+c)^2[-bc + ac + c^2 + ab + b^2 - bc - ab + a^2 + ac - b^2 + ab + bc + bc -$

$-ac - c^2 \quad] = 2bc(a+b+c)^2[a^2 + ab + ac] = 2abc(a+b+c)^3.$

Answer. $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$