

Answer on Question #44408 – Math – Linear Algebra

Question. Use the properties of determinants to evaluate the following determinant:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}.$$

Solution. Multiply the second column by -1 and add it to the third column, the result is written to the third column:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = \begin{vmatrix} (b+c)^2 & a^2 & 0 \\ b^2 & (c+a)^2 & (a+b+c)(b-a-c) \\ c^2 & c^2 & (a+b+c)(a+b-c) \end{vmatrix}. \quad \text{Further}$$

multiply the first column by -1 and add it to the second column, the result is written to the second column :

$$\begin{vmatrix} (b+c)^2 & a^2 & 0 \\ b^2 & (c+a)^2 & (a+b+c)(b-a-c) \\ c^2 & c^2 & (a+b+c)(a+b-c) \end{vmatrix} =$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & 0 \\ b^2 & (a+b+c)(a+c-b) & (a+b+c)(b-a-c) \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}. \quad \text{Take the common factor}$$

$a+b+c$ from the second column and from the third column. We shall have

$$\begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & 0 \\ b^2 & (a+b+c)(a+c-b) & (a+b+c)(b-a-c) \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix} =$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & a+c-b & b-a-c \\ c^2 & 0 & a+b-c \end{vmatrix}. \quad \text{In the last determinant add the third}$$

column to the second column:

$$(a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & a+c-b & b-a-c \\ c^2 & 0 & a+b-c \end{vmatrix} =$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & a+b-c \end{vmatrix}. \quad \text{Further add the second row to the third}$$

row:

$$(a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & 0 \\ b^2 & 0 & b-a-c \\ c^2 & a+b-c & a+b-c \end{vmatrix} =$$

$$= (a + b + c)^2 \begin{vmatrix} (b + c)^2 & a - b - c & 0 \\ b^2 & 0 & b - a - c \\ b^2 + c^2 & a + b - c & 2b - 2c \end{vmatrix}. \text{ Multiply the third row by } -1 \text{ and add it to the first row. Further take the common factor 2 from the first row:}$$

$$(a + b + c)^2 \begin{vmatrix} (b + c)^2 & a - b - c & 0 \\ b^2 & 0 & b - a - c \\ b^2 + c^2 & a + b - c & 2b - 2c \end{vmatrix} =$$

$$= (a + b + c)^2 \begin{vmatrix} 2bc & -2b & 2c - 2b \\ b^2 & 0 & b - a - c \\ b^2 + c^2 & a + b - c & 2b - 2c \end{vmatrix} =$$

$$= 2(a + b + c)^2 \begin{vmatrix} bc & -b & c - b \\ b^2 & 0 & b - a - c \\ b^2 + c^2 & a + b - c & 2b - 2c \end{vmatrix}. \text{ Multiply the second row by } -1 \text{ and add it to the third row:}$$

$$2(a + b + c)^2 \begin{vmatrix} bc & -b & c - b \\ b^2 & 0 & b - a - c \\ b^2 + c^2 & a + b - c & 2b - 2c \end{vmatrix} =$$

$$= 2(a + b + c)^2 \begin{vmatrix} bc & -b & c - b \\ b^2 & 0 & b - a - c \\ c^2 & a + b - c & a + b - c \end{vmatrix}. \text{ Multiply the second column by } -1 \text{ and add it to the third column:}$$

$$2(a + b + c)^2 \begin{vmatrix} bc & -b & c - b \\ b^2 & 0 & b - a - c \\ c^2 & a + b - c & a + b - c \end{vmatrix} =$$

$$= 2(a + b + c)^2 \begin{vmatrix} bc & -b & c \\ b^2 & 0 & b - a - c \\ c^2 & a + b - c & 0 \end{vmatrix}. \text{ We shall evaluate the last determinant using the rule of triangle:}$$

$$2(a + b + c)^2 \begin{vmatrix} bc & -b & c \\ b^2 & 0 & b - a - c \\ c^2 & a + b - c & 0 \end{vmatrix} =$$

$$= 2(a + b + c)^2 [-bc^2(b - a - c) + b^2c(a + b - c) - bc(a + b - c)(b - a - c)] =$$

$$= 2bc(a + b + c)^2 [-bc + ac + c^2 + ab + b^2 - bc - ab + a^2 + ac - b^2 + ab + bc + bc - ac - c^2] = 2bc(a + b + c)^2 [a^2 + ab + ac] = 2abc(a + b + c)^3.$$

$$\text{Answer. } \begin{vmatrix} (b + c)^2 & a^2 & a^2 \\ b^2 & (c + a)^2 & b^2 \\ c^2 & c^2 & (a + b)^2 \end{vmatrix} = 2abc(a + b + c)^3.$$