Answer on Question 44405 - Math - Linear Algebra

$$B_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

be 2 bases for $\operatorname{span} B_1 \subset M_{2x2}(\mathbb{R})$ with the usual left to right ordering. Let B_3 be a basis for P_1 and be a transition matrix from B_2 to B_3 given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = PB_2 \to B_3$$

(a) Find transition matrix $PB_1 \to B_3$ (b) Use $PB_2 \to B_3$ to find B_3

Firstly, we notice that bases B_1 and B_2 are bases for linear subspace

$$B = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, x, y, z \in \mathbb{R} \right\}$$

which is isomorphic to $\mathbb{R}^3$. Let's find respective bases of B in $\mathbb{R}^3$ for B_1 and B_2
According to B_1

$$B = \begin{bmatrix} \alpha+\beta+\gamma & \alpha \\ 0 & \alpha+\beta-\gamma \end{bmatrix}, \alpha, \beta, \gamma \in \mathbb{R}$$

that's why it has a base

$$PB_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Analogically,

$$PB_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

It's known that

$$B_3 \to PB_2 = (PB_2 \to B_3)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, we can find B_3 as a matrix $3x3$

$$B_3 = B_3 \to PB_2 \cdot PB_2 \cdot PB_2 \to B_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

and B_3 as a span of $2x2$ matrices

$$B_3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \right\}$$

Due to standart theory of transition matrix

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = s_{11} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s_{21} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_{31} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = s_{12} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s_{22} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_{32} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = s_{13} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s_{23} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s_{33} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

It's easy to show that

$$s_{11} = 0; \quad s_{21} = \frac{1}{2}; \quad s_{31} = \frac{1}{2}$$

$$s_{12} = 0; \quad s_{22} = \frac{1}{2}; \quad s_{32} = \frac{-1}{2}$$

$$s_{13} = 2; \quad s_{23} = \frac{-5}{2}; \quad s_{33} = \frac{-1}{2}$$

So,$S = PB_1 \to B_3 =$

$$\begin{bmatrix} 0 & 0 & 2 \\ \frac{1}{2} & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}$$