Answer on Question# #44404 – Math – Real Analysis

Question:

Determine if the series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{m=1}^{\infty} \frac{\sin\left(\frac{(2m+1)\pi}{2}\right)}{m+1} \ln(m+1) \quad (1)$$

Solution:

Let's simplify the expression $\sin\left(\frac{(2m+1)\pi}{2}\right)$ in (1):

$$\sin\left(\frac{(2m+1)\pi}{2}\right) = \sin\left(\frac{\pi}{2} + \pi m\right) = \sin\left(\frac{\pi}{2}\right)\cos(\pi m) + \cos\left(\frac{\pi}{2}\right)\sin(\pi m) = \cos(\pi m) = (-1)^{\mathrm{m}}.$$

Then series (1) can be rewritten as

$$\sum_{m=1}^{\infty} \frac{(-1)^{\mathrm{m}}}{m+1} \ln(m+1) \quad (1a)$$

As we see, the series (1) is alternating series. By definition, the alternating series $\sum_{m=1}^{\infty} a_m$ converges absolutely if the series $\sum_{m=1}^{\infty} |a_m|$ converges. A series converges conditionally if it converges but does not converge absolutely.

The series (1a) by the Leibniz test is convergent, as $\lim_{m \to \infty} \frac{\ln(m+1)}{m+1} = 0$. Let's compose the absolute value series

$$\sum_{m=1}^{\infty} \frac{\ln(m+1)}{m+1}. \quad (2)$$

To determine, whether the series (2) converges or diverges, we use the comparison test. Let's consider following series

$$\sum_{m=1}^{\infty} \frac{1}{m}. \quad (3)$$

For all $m \geq 4$ inequality $\frac{\ln(m+1)}{m+1} > \frac{1}{m}$ takes place. The series (3) is the harmonic series, which diverges. Therefore, the absolute value series (2) diverges by comparison.

Hence, the original series is conditionally convergent.

Answer: The series $\sum_{m=1}^{\infty} \frac{\sin\left(\frac{(2m+1)\pi}{2}\right)}{m+1} \ln(m+1)$ is conditionally convergent.