Answer on Question #44366 – Math - Abstract Algebra

Problem.

Which of the following statements are true and which are false? Justify your answer with a short proof or a counterexample.
i) On the set f1;2;3g, R = f(1;1); (2;2); (3;3)g is an equivalence relation.
ii) No non-abelian group of order n can have an element of order n.
iii) For every composite natural number n, there is a non-abelian group of order n.
iv) Every Sylow p-subgroup of a finite group is normal.
v) If a commutative ring with unity has zero divisors, it also has nilpotent elements.
vi) If R is a ring with identity and u 2 R is a unit in R, 1+u is not a unit in R.
vii) If a and b are elements of a group G such that o(a) = 2, o(b) = 3, then o(ab) = 6.
viii) Every integral domain is an Euclidean domain.
ix) The quotient field of the ring
fa+ibja;b 2 Zg
is C.
x) The field Q(p2) is not the subfield of any field of characteristic p, where p > 1 is a prime.
Remark. The statement of the problem is formatted incorrectly. I suppose that the correct statement is
"Which of the following statements are true and which are false? Justify your answer with a short proof or a counterexample.
i) On the set $\{1,2,3\}$, $R = \{(1;1),(2;2),(3;3)\}$ is an equivalence relation.
ii) No non-abelian group of order n can have an element of order n.
iii) For every composite natural number n, there is a non-abelian group of order n.
iv) Every Sylow p-subgroup of a finite group is normal.
v) If a commutative ring with unity has zero divisors, it also has nilpotent elements.
vi) If R is a ring with identity and $u \in R$ is a unit in R, $1+u$ is not a unit in R.
vii) If a and b are elements of a group G such that $o(a) = 2$, $o(b) = 3$, then $o(ab) = 6$.
viii) Every integral domain is an Euclidean domain.
ix) The quotient field of the ring

$$\{a+ib|a,b \in \mathbb{Z}\}$$

is $\mathbb{C}$.
x) The field $\mathbb{Q}(\sqrt{2})$ is not the subfield of any field of characteristic p, where $p > 1$ is a prime."

Solution.

i) True.
The reflexivity holds, as for all $\in \{1,2,3\}$ $(x,x) \in R$.
The symmetry holds, as if $(x,y) \in R$, then $(y,x) \in R$, as there are only elements of type (x,x) in R.
The transitivity holds, as if $(x,y) \in R$, $(y,z) \in R$, then $(x,z) \in R$, as there are only elements of type (x,x) in R.

ii) True.
If there is element of order n in the group of order n, then this group is cyclic. The cyclic group is abelian.

iii) False.
Each group of order 4 is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

iv) False.

The S_4 doesn't have normal Sylow subgroup.

v) False.
$\mathbb{Z}_6$ has zero divisor, but doesn't have nilpotent elements.

vi) False.
If $R = \mathbb{R}$ and $u = 2$, then $u = 3$ is unit.

vii) False.
If $a = (1\ 2) \in S_3$ and $b = (1\ 2\ 3) \in S_3$, then $ab = (3\ 2) \in S_3$ and $o\big((3\ 2)\big) = 2$.

viii) False.
$\mathbb{Q}(\sqrt{-19})$ is integral domain, but isn't an Euclidean domain.

ix) False.
The quotient field is field with element (x, y) where $x, y \in R$ and R should be an integral domain.

x) True.
If field $\mathbb{Q}(\sqrt{2})$ is the subfield of any field of characteristic p, then $\sqrt{2}p = 0$.