Answer on Question #44363 - Math - Abstract Algebra

Question *Find, with justification, all the Sylow subgroups of* Z_{15}.

Solution. The group Z_{15} has order 15=3·5, therefore its Sylow subgroups has orders 3 or 5. Since the numbers 3 and 5 are prime, the groups of order 3 or 5 are cyclic. Hence, the desired subgroups are generated by the elements of Z_{15} of order 3 or 5. We know that the order $|n|$ of an element $n \in Z_{15}$ equals $n/gcd(n,15)$. So, $|n|=3$ if and only if $gcd(n,15)=5$, while $|n|=5$ if and only if $gcd(n,15)=3$. The numbers n between 0 and 14, such that $gcd(n,15)=5$, are

$$n=5,10.$$

Since 10=5·2, the element 10 generates the same cyclic subgroup as 5 does. Thus, Z_{15} has a unique subgroup of order 3. It is $\langle 5\rangle$.

The numbers n between 0 and 14, such that $gcd(n,15)=3$, are

$$n=3,6,9,12.$$

All these numbers belong to the same cyclic subgroup $\langle 3\rangle$ of order 5. Thus, Z_{15} has a unique subgroup of order 5. It is $\langle 3\rangle$.

Answer: $\langle 5\rangle=\{0,5,10\}$ of order 3 and $\langle 3\rangle=\{0,3,6,9,12\}$ of order 5.