Answer on Question #44359 – Math - Abstract Algebra

Problem.

Let G = S4, H = A4 and K = f1; (1 2)(3 4); (1 3)(2 4); (1 4)(2 3)g.
i) Check that H=K = h(1 2 3)Hi
ii) Check that K is normal in H.(Hint: For each h 2 H,h 62 K, check that hK = Kh.)
iii) Check whether (1 2 3 4))H is the inverse of (1 3 4 2)H in the group S4=H.

Remark.

The statement isn't correctly formatted. I suppose that the correct statement is
"Let $G = S_4, H = A_4$ and $K = \{1;\ (1\,2)(3\,4);\ (1\,3)(2\,4);\ (1\,4)(2\,3)\}$.
i) Check that $H/K = \langle(1\,2\,3)H\rangle$.
ii) Check that K is normal in H.(Hint: For each $h \in H, h \notin K$, check that $hK = Kh$.)
iii) Check whether $(1\,2\,3\,4)H$ is the inverse of $(1\,3\,4\,2)H$ in the group S_4/H."

Solution.

$$H = \left\{\begin{matrix} 1; (1\,2)(3\,4);\ (1\,3)(2\,4);\ (1\,4)(2\,3); (1\,2\,3); (1\,2\,4);\ (1\,3\,2);\ (1\,3\,4);\ (1\,4\,2); \\ (1\,4\,3);\ (2\,3\,4);\ (2\,4\,3) \end{matrix}\right\}.$$

i) $$(1\,2\,3)H = \left\{\begin{matrix} (1\,2\,3); (1\,2\,3)(1\,2)(3\,4);\ (1\,2\,3)(1\,3)(2\,4);\ (1\,2\,3)(1\,4)(2\,3); \\ (1\,2\,3)(1\,2\,3); (1\,2\,3)(1\,2\,4);\ (1\,2\,3)(1\,3\,2);\ (1\,2\,3)(1\,3\,4);\ (1\,2\,3)(1\,4\,2); \\ (1\,2\,3)(1\,4\,3); (1\,2\,3)(2\,3\,4);\ (1\,2\,3)(2\,4\,3) \end{matrix}\right\} =$$
$\{(1\,2\,3); (1\,3\,4); (2\,4\,3); (1\,4\,2); (1\,3\,2); \dots\}$
There are no class in H/K that has 5 different elements (it has at most 4 elements).

ii) We will check that for all$h \in H, h \notin K$, check that $hK = Kh$:

- $(1\,2\,3)H = \{(1\,2\,3);\ (1\,2\,3)(1\,2)(3\,4);\ (1\,2\,3)(1\,3)(2\,4);\ (1\,2\,3)(1\,4)(2\,3)\} = \{(1\,2\,3); (1\,3\,4); (2\,4\,3); (1\,4\,2)\}$ and
  $H(1\,2\,3) = \{(1\,2\,3);\ (1\,2)(3\,4)(1\,2\,3);\ (1\,3)(2\,4)(1\,2\,3);\ (1\,4)(2\,3)(1\,2\,3)\} = \{(1\,2\,3); (2\,4\,3); (1\,4\,2); (1\,3\,4)\}$.
  Hence $(1\,2\,3)H = H(1\,2\,3)$.
- $(1\,2\,4)H = \{(1\,2\,4);\ (1\,2\,4)(1\,2)(3\,4);\ (1\,2\,4)(1\,3)(2\,4);\ (1\,2\,4)(1\,4)(2\,3)\} = \{(1\,2\,4); (1\,4\,3); (1\,3\,2); (2\,3\,4)\}$ and
  $H(1\,2\,4) = \{(1\,2\,4);\ (1\,2)(3\,4)(1\,2\,4);\ (1\,3)(2\,4)(1\,2\,4);\ (1\,4)(2\,3)(1\,2\,4)\} = \{(1\,2\,4); (2\,3\,4); (1\,4\,3); (1\,3\,2)\}$.
  Hence $(1\,2\,4)H = H(1\,2\,4)$.
- $(1\,3\,2)H = \{(1\,3\,2);\ (1\,3\,2)(1\,2)(3\,4);\ (1\,3\,2)(1\,3)(2\,4);\ (1\,3\,2)(1\,4)(2\,3)\} = \{(1\,3\,2); (2\,3\,4); (1\,2\,4); (1\,4\,3)\}$ and
  $H(1\,3\,2) = \{(1\,3\,2);\ (1\,2)(3\,4)(1\,3\,2);\ (1\,3)(2\,4)(1\,3\,2);\ (1\,4)(2\,3)(1\,3\,2)\} = \{(1\,3\,2); (1\,4\,3); (2\,3\,4); (1\,2\,4)\}$.
  Hence $(1\,3\,2)H = H(1\,3\,2)$.
- $(1\,3\,4)H = \{(1\,3\,4);\ (1\,3\,4)(1\,2)(3\,4);\ (1\,3\,4)(1\,3)(2\,4);\ (1\,3\,4)(1\,4)(2\,3)\} = \{(1\,3\,4); (1\,2\,3); (1\,4\,2); (2\,4\,3)\}$ and
  $H(1\,3\,4) = \{(1\,3\,4);\ (1\,2)(3\,4)(1\,3\,4);\ (1\,3)(2\,4)(1\,3\,4);\ (1\,4)(2\,3)(1\,3\,4)\} = \{(1\,3\,4); (1\,4\,2); (2\,4\,3); (1\,2\,3)\}$.
  Hence $(1\,3\,4)H = H(1\,3\,4)$.
- $(1\,4\,2)H = \{(1\,4\,2);\ (1\,4\,2)(1\,2)(3\,4);\ (1\,4\,2)(1\,3)(2\,4);\ (1\,4\,2)(1\,4)(2\,3)\} = \{(1\,4\,2); (2\,4\,3); (1\,3\,4); (1\,2\,3)\}$ and
  $H(1\,4\,2) = \{(1\,4\,2);\ (1\,2)(3\,4)(1\,4\,2);\ (1\,3)(2\,4)(1\,4\,2);\ (1\,4)(2\,3)(1\,4\,2)\} = \{(1\,4\,2); (1\,3\,4); (1\,2\,3); (2\,4\,3)\}$.
  Hence $(1\,4\,2)H = H(1\,4\,2)$.
- $(1\,4\,3)H = \{(1\,4\,3);\ (1\,4\,3)(1\,2)(3\,4);\ (1\,4\,3)(1\,3)(2\,4);\ (1\,4\,3)(1\,4)(2\,3)\} = \{(1\,4\,3); (1\,2\,4); (2\,3\,4); (1\,3\,2)\}$ and

$H(1\ 4\ 3) = \{(1\ 4\ 3);\ (1\ 2)(3\ 4)(1\ 4\ 3);\ (1\ 3)(2\ 4)(1\ 4\ 3);\ (1\ 4)(2\ 3)(1\ 4\ 3)\} = \{(1\ 4\ 3); (1\ 3\ 2); (1\ 2\ 4); (2\ 3\ 4)\}$.
Hence $(1\ 4\ 3)H = H(1\ 4\ 3)$.

- $(2\ 3\ 4)H = \{(2\ 3\ 4);\ (2\ 3\ 4)(1\ 2)(3\ 4);\ (2\ 3\ 4)(1\ 3)(2\ 4);\ (2\ 3\ 4)(1\ 4)(2\ 3)\} = \{(2\ 3\ 4); (1\ 3\ 2); (1\ 4\ 3); (1\ 2\ 4)\}$ and
  $H(2\ 3\ 4) = \{(2\ 3\ 4);\ (1\ 2)(3\ 4)(2\ 3\ 4);\ (1\ 3)(2\ 4)(2\ 3\ 4);\ (1\ 4)(2\ 3)(2\ 3\ 4)\} = \{(2\ 3\ 4); (1\ 2\ 4); (1\ 3\ 2); (1\ 4\ 3)\}$.
  Hence $(2\ 3\ 4)H = H(2\ 3\ 4)$.
- $(2\ 4\ 3)H = \{(2\ 4\ 3);\ (2\ 4\ 3)(1\ 2)(3\ 4);\ (2\ 4\ 3)(1\ 3)(2\ 4);\ (2\ 4\ 3)(1\ 4)(2\ 3)\} = \{(2\ 4\ 3); (1\ 4\ 2); (1\ 2\ 3); (1\ 3\ 4)\}$ and
  $H(2\ 4\ 3) = \{(2\ 4\ 3);\ (1\ 2)(3\ 4)(2\ 4\ 3);\ (1\ 3)(2\ 4)(2\ 4\ 3);\ (1\ 4)(2\ 3)(2\ 4\ 3)\} = \{(2\ 4\ 3); (1\ 2\ 3); (1\ 3\ 4); (1\ 4\ 2)\}$.
  Hence $(2\ 4\ 3)H = H(2\ 4\ 3)$.

Therefore K is normal in H.
iii) $(1\ 2\ 3\ 4)H(1\ 3\ 4\ 2)H = (1\ 2\ 3\ 4)(1\ 3\ 4\ 2)H = (1\ 4\ 3)H \neq H$. Therefore $(1\ 2\ 3\ 4)H$ isn't an inverse to $(1\ 3\ 4\ 2)H$.