Answer on Question #44251 – Math - Differential Calculus | Equations

Solve the following Cauchy Euler equation by the method of variation of parameters.

$$x^2y^n - xy' + y = 2x$$

Determine the singular points of the following differential equation and classify each singular point as regular or irregular.

$$(x^2 - 9)^2 y^n + (x + 3) y' + 2y = 0.$$

Solution.

Solve the Cauchy Euler equation by the method of variation of parameters.

By definition a Cauchy–Euler equation of order n has the form

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) \ldots + a_0 y(x) = 0$$

So the correct equation

$$x^2y'' - xy' + y = 2x$$

Solve this equation. The general solution will be the sum of the complementary solution and particular solution.

Find the complementary solution by solving $x^2y'' - xy' + y = 0$.

Assume a solution to this Euler-Cauchy equation will be proportional to x^λ for some constant λ.

Substitute $y(x) = x^\lambda$ into the differential equation:

$$x^2(x^\lambda)'' - x(x^\lambda)' + x^\lambda = 0$$

$$\lambda^2 x^\lambda - 2\lambda x^\lambda + x^\lambda = 0$$

Factor out x^λ:

$$(\lambda^2 - 2\lambda + 1)x^\lambda = 0$$

Assuming $x \neq 0$, the zeros must come from the polynomial:

$$\lambda^2 - 2\lambda + 1 = 0$$

Solve for λ:

$$\lambda_{1,2} = 1$$

The multiplicity of the root $\lambda = 1$ is 2 which gives $y_1(x) = c_1 x,\ y_2(x) = c_2 x \ln x$ as solutions, where c_1 and c_2 are arbitrary constants.

The complementary solution is the sum of the above solutions:

$$y_c(x) = y_1(x) + y_2(x) = c_1 x + c_2 x \ln x$$

Determine the particular solution by variation of parameters.

List the basis solutions in $y_c(x)$:

$$y_{b_1}(x) = x \text{ and } y_{b_2}(x) = x \ln x$$

Compute the Wronskian of $y_{b_1}(x)$ and $y_{b_2}(x)$:

$$W(x) = \begin{vmatrix} x & x \ln x \\ \frac{d}{dx}(x) & \frac{d}{dx}(x \ln x) \end{vmatrix} = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x$$

Divide the differential equation by the leading term's coefficient x^2:

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = \frac{2}{x}$$

Let $f(x) = \frac{2}{x}$. Let $v_1(x) = -\int \frac{f(x) y_{b_2}(x)}{W(x)} dx$ and $v_2(x) = -\int \frac{f(x) y_{b_1}(x)}{W(x)} dx$.

The particular solution will be given by:

$$y_p(x) = v_1(x) y_{b_1}(x) + v_2(x) y_{b_2}(x)$$

Compute $v_1(x)$:

$$v_1(x) = -\int \frac{2 \ln x}{x} dx = -\ln^2 x$$

Compute $v_2(x)$:

$$v_2(x) = \int \frac{2}{x} dx = 2 \ln x$$

The particular solution is thus:

$$y_p(x) = v_1(x) y_{b_1}(x) + v_2(x) y_{b_2}(x) = -x \ln^2 x + 2x \ln^2 x$$

Simplify:

$$y_p(x) = x \ln^2 x$$

So the general solution:

$$y(x) = y_c(x) + y_p(x) = c_1 x + c_2 x \ln x + x \ln^2 x$$

Answer: $y(x) = c_1 x + c_2 x \ln x + x \ln^2 x$.

Determine the singular points of the following differential equation and classify each singular point as regular or irregular.

Equation in standard form

$$N(x)y'' + P(x)y' + Q(x)y = 0$$

So our equation

$$(x^2 - 9)^2 y'' + (x + 3)y' + 2y = 0$$

The singular points of the equation are then the points where $N(x) = 0$:

$$(x^2 - 9)^2 = 0$$

$$x_{1,2} = 3, x_{3,4} = -3$$

Suppose x_0 is a singular point. Multiplying through by $(x - x_0)^2/N(x)$, we may rewrite equation as

$$(x - x_0)^2 y'' + (x - x_0)u(x)y' + v(x)y = 0,$$

where

$$u(x) = \frac{(x - x_0)P(x)}{N(x)}, \quad v(x) = \frac{(x - x_0)^2 Q(x)}{N(x)},$$

We say that x_0 is a regular singular point if the rational functions $u(x)$ and $v(x)$ have no singularity at x_0 – that is, if the factors of $x - x_0$ in $N(x)$ that cause $N(x)$ to vanish at x_0 are canceled by such factors in $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$. Otherwise x_0 is an irregular singular point.

Consider $x_0 = 3$. Find $u(x)$ and $v(x)$:

$$u(x) = \frac{(x - 3)(x + 3)}{(x^2 - 9)^2} = \frac{1}{x^2 - 9}$$

$$v(x) = \frac{(x - 3)^2 \cdot 2}{(x^2 - 9)^2} = \frac{2}{(x + 3)^2}$$

$u(x)$ has singularity at x_0, so $x_0 = 3$ is an irregular singular point.

Consider $x_0 = -3$. Find $u(x)$ and $v(x)$:

$$u(x) = \frac{(x + 3)(x + 3)}{(x^2 - 9)^2} = \frac{1}{(x - 3)^2}$$

$$v(x) = \frac{(x + 3)^2 \cdot 2}{(x^2 - 9)^2} = \frac{2}{(x - 3)^2}$$

$u(x)$ and $v(x)$ have no singularity at x_0, so $x_0 = -3$ is a regular singular point.

Answer: $x_0 = 3$ is an **irregular** singular point, $x_0 = -3$ is a **regular** singular point.