Answer on Question #44086 – Math - Complex Analysis

Problem.

If 〖(1+x+x^2)〗^n = a_0+a_2 x^2+ ------ +a_2n x^2n then prove that a_0+a_3+a_6+ ----- = 3^(n-1).

Remark.

I suppose that part of question is missed. The correct statement is
"If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, then prove that $a_0 + a_3 + a_6 + \cdots = 3^{n-1}$".
as polynomial $(1+x+x^2)^n$ has a monomial $2x$.

Solution:

The numbers $\xi = e^{\frac{2\pi i}{3}}$ and $\xi^2 = e^{\frac{4\pi i}{3}}$ are the primitive third roots of unity. Therefore they are the roots of the polynomial $1 + x + x^2$ (as $x^3 - 1 = (x-1)(x^2+x+1)$) and $\xi^3 = 1$.
If we put $x \to \xi$ into the equation

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

we will obtain

$$(1+\xi+\xi^2)^n = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_{2n}\xi^{2n}$$

or

$$0 = a_0 + a_1\xi + a_2\xi^2 + a_3 + a_4\xi + a_5\xi^2 + \cdots, \quad (1)$$

as $1 + \xi + \xi^2 = 0$ and $\xi^{3k} = 1$ for all integer k.
If we put $x \to \xi^2$ into the equation

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

we will obtain

$$(1+\xi^2+\xi^4)^n = a_0 + a_1\xi^2 + a_2\xi^4 + \cdots + a_{2n}\xi^{4n}$$

or

$$0 = a_0 + a_1\xi^2 + a_2\xi + a_3 + a_4\xi^2 + a_5\xi + \cdots, \quad (2)$$

as $1 + \xi^2 + \xi^4 = 1 + \xi^2 + \xi = 0$ and $\xi^{3k} = 1$ for all integer k.
If we put $x \to 1$ into the equation

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

we will obtain

$$(1+1+1)^n = a_0 + a_1 + a_2 + \cdots + a_{2n}$$

or

$$3^n = a_0 + a_1 + a_2 + \cdots + a_{2n} \quad (3)$$

If we add (1), (2) and (3) equations we will obtain

$$3^n = 3a_0 + (1+\xi+\xi^2)a_1 + (1+\xi+\xi^2)a_2 + 3a_3 + (1+\xi+\xi^2)a_4 + (1+\xi+\xi^2)a_5 + \cdots$$

or

$$3^n = 3a_0 + 3a_3 + \cdots,$$

as $1 + \xi + \xi^2 = 0$.
Hence

$$3^{n-1} = a_0 + a_3 + \cdots.$$