Answer on Question #44085 – Math - Complex Analysis

If x = a + b, y = aα + bβ and z = aβ + bα where α and β are complex cube roots of unity.

Show that $xyz = a^3 + b^3$.

Solution

Since, α, β are the complex cube roots of unity. We take $\alpha = \omega$ and $\beta = \omega^2$. Now,

$$xyz = (\mathrm{a}+\mathrm{b})(\mathrm{a}\alpha+\mathrm{b}\beta)(\mathrm{a}\beta+\mathrm{b}\alpha) = (\mathrm{a}+\mathrm{b})[a^2\alpha\beta+\mathrm{ab}(\alpha^2+\beta^2)+\mathrm{b}^2\alpha\beta]$$
$$= (\mathrm{a}+\mathrm{b})[a^2(\omega\cdot\omega^2)+\mathrm{ab}(\omega^2+\omega^4)+\mathrm{b}^2(\omega\cdot\omega^2)].$$

But

$$\omega\cdot\omega^2 = \omega^3 = 1$$

and

$$\omega^2+\omega^4 = \omega^2+\omega\cdot\omega^3 = \omega^2+\omega\cdot 1 = \omega^2+\omega.$$

Since

$$\omega^2+\omega+1 = 0 \rightarrow \omega^2+\omega = -1.$$

So,

$$xyz = (\mathrm{a}+\mathrm{b})[a^2-\mathrm{ab}+\mathrm{b}^2] = a^3+b^3.$$