Answer on Question #43904 – Math – Other

Prove that $1903 < \left(3+\sqrt{13}\right)^4 < 1904$.

Solution.

Firstly, compute $\left(3+\sqrt{13}\right)^4$:

$$\left(3+\sqrt{13}\right)^4 = \left(9+2\cdot 3\cdot\sqrt{13}+13\right)^2 = 4\cdot\left(11+3\sqrt{13}\right)^2 =$$

$$= 4\cdot\left(121+2\cdot 11\cdot 3\sqrt{13}+9\cdot 13\right) = 4\cdot\left(238+66\sqrt{13}\right) = 952+264\sqrt{13};$$

Now simplify our double inequality:

$$1903 < \left(3+\sqrt{13}\right)^4 < 1904 \Leftrightarrow 951 < 264\sqrt{13} < 952 \Leftrightarrow \frac{951}{264} < \sqrt{13} < \frac{952}{264} \Leftrightarrow$$

$$\Leftrightarrow \frac{317}{88} < \sqrt{13} < \frac{119}{33} \Leftrightarrow 3\frac{53}{88} < \sqrt{13} < 3\frac{20}{33} \Leftrightarrow \left(3+\frac{53}{88}\right)^2 < 13 < \left(3+\frac{20}{33}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow 9+6\cdot\frac{53}{88}+\frac{53^2}{88^2} < 13 < 9+6\cdot\frac{20}{33}+\frac{20^2}{33^2} \Leftrightarrow \frac{27}{44}+\frac{53^2}{88^2} < 1 < \frac{7}{11}+\frac{20^2}{33^2};$$

Consider these equations separately:

$$\frac{27}{44}+\frac{53^2}{88^2} < 1 \Leftrightarrow \frac{53^2}{88^2} < \frac{17}{44} \Leftrightarrow 53^2 < 88\cdot 34 \Leftrightarrow 2809 < 2992 - \text{it's true};$$

$$1 < \frac{7}{11}+\frac{20^2}{33^2} \Leftrightarrow \frac{4}{11} < \frac{400}{33^2} \Leftrightarrow 33\cdot 12 < 400 \Leftrightarrow 396 < 400 - \text{it's true};$$

Hence, both inequalities are true, so the whole double inequality is true.