Answer on Question #43852 - Math – Differential Calculus | Equations

Calculate F_{xxyz} if $F(x,y,z) = \sin(3x + yz)$.

Solution

Let $F(x,y,z) = \sin(3x + yz)$.

$$F_{xxyz} = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}F(x,y,z) = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\sin(3x + yz).$$

Calculate step by step four partial derivatives.

First, find the partial derivative of a given function with respect to x. To calculate $\frac{\partial}{\partial x}F(x,y,z)$, we simply view y and z as being a fixed number and calculate the ordinary derivative with respect to x. In calculating partial derivatives, we can use all the rules for ordinary derivatives.

Recall the derivative of the sine function: $\frac{d}{dx}\sin(x) = \cos(x)$

We can think of continuous function $F(x,y,z) = \sin(3x + yz)$ as being the result of combining two continuous functions. If $g(x) = 3x + yz$ is the inside function and $h(t) = \sin(t)$ is the outside function, then the result of substituting of $g(x)$ into the function h is

$$h(g(x)) = \sin(3x + yz)$$

By the chain rule, if $f(x) = h(g(x))$, then $(f)'_x(x) = (h)'_x(g(x)) \times (g)'_x(x)$.

Here $(\sin(3x + yz))'_x = (\sin)'_x(3x + yz) \times (3x + yz)'_x = \cos(3x + yz) \times (3x + yz)'_x$.

The sum rule tells us that for two functions u and v :

$$(u + v)'_x(x) = (u)'_x(x) + (v)'_x(x)$$

Let $u(x) = 3x$, $v(x) = yz$. We view function $v(x) = yz$ being a constant with respect to x. The derivative of constant c is zero: $(c)'_x = \frac{d}{dx}c = 0$. Here $(yz)'_x = \frac{\partial}{\partial x}(yz) = 0$.

The derivative of a constant multiplied by a function is the constant multiplied by the derivative of the original function:
$(af(x))'_x = a(f(x))'_x$.

Here $(3x)'_x = 3(x)'_x = 3 \times 1 = 3$.

Therefore, $(3x + yz)'_x = (3x)'_x + (yz)'_x = 3 + 0 = 3$.

So,

$$F_x = \frac{\partial}{\partial x}F(x,y,z) = \frac{\partial}{\partial x}\sin(3x + yz) = \frac{d}{dt}\sin(t)\Big|_{t=3x+yz} \times (3x + yz)'_x = \cos(3x + yz) \times 3 =$$

$= 3\cos(3x + yz)$.

Next, recall the derivative of the cosine function: $\frac{d}{dx}\cos(x) = -\sin(x)$.

So,

$$F_{xx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}F(x,y,z) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\sin(3x+yz) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\sin(3x+yz))\right) = \frac{\partial}{\partial x}(3\cos(3x+yz)) =$$

$$= 3\frac{\partial}{\partial x}(\cos(3x+yz)) = 3\frac{d}{dt}\cos(t)\bigg|_{t=3x+yz} \times (3x+yz)'_x = -3\sin(3x+yz)\times 3 =$$

$$= -9\sin(3x+yz).$$

Next,

$$F_{xxy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}F(x,y,z) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\sin(3x+yz) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\sin(3x+yz)\right) = \frac{\partial}{\partial y}\left(-9\sin(3x+yz)\right) =$$

$$= -9\frac{\partial}{\partial y}(\sin(3x+yz)) = -9\frac{d}{dt}\sin(t)\bigg|_{t=3x+yz} \times (3x+yz)'_y = -9\cos(3x+yz)\times(0+z) =$$

$$= -9z\cos(3x+yz).$$

Next,

$$F_{xxyz} = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}F(x,y,z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}F(x,y,z)\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\sin(3x+yz)\right) =$$

$$= \frac{\partial}{\partial z}\left(-9z\cos(3x+yz)\right) = -9\frac{\partial}{\partial z}\left(z\cos(3x+yz)\right) = -9\cos(3x+yz)\times\frac{\partial z}{\partial z} - 9z\frac{\partial}{\partial z}\left(\cos(3x+yz)\right) =$$

$$= -9\cos(3x+yz)\times 1 - 9z\frac{d}{dt}\cos(t)\bigg|_{t=3x+yz} \times (3x+yz)'_z =$$

$$= -9\cos(3x+yz) - 9z\times(-\sin(t))|_{t=3x+yz}\times(0+y) =$$

$$= -9\cos(3x+yz) + 9z\sin(3x+yz)\times y =$$

$$= -9\cos(3x+yz) + 9yz\sin(3x+yz).$$

Note, If the mixed partial derivatives exist and are continuous at a point x_0, then they are equal at x_0 regardless of the order in which they are taken.

Answer: $F_{xxyz} = -9\cos(3x+yz) + 9yz\sin(3x+yz)$