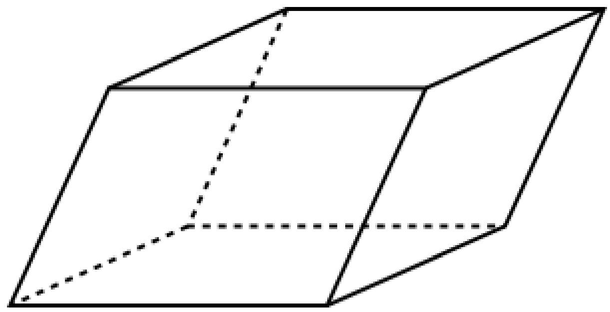
Answer on Question #43544 – Math – Geometry

Prove that in a parallelepiped the sum of the diagonals is equal to the sum of the squares on the edges.

Proof.

A parallelepiped is a three-dimensional geometric solid with six faces that are parallelograms.



Parallelograms which is composed of a parallelepiped, called its faces, their side - ribs, and the vertices of parallelograms - tops the parallelepiped.

Based on the definition of a parallelepiped, we know that the sides are parallelograms, and hence in the proof will be guided with parallelogram properties.

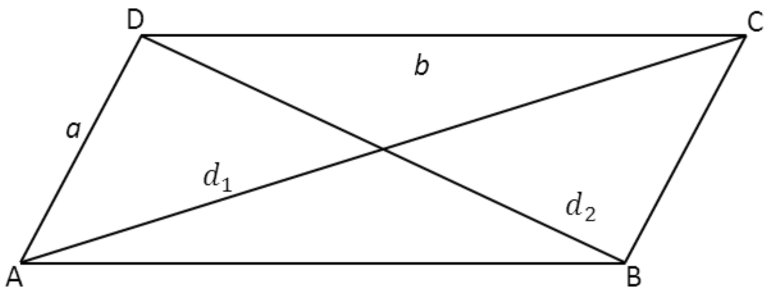
We solve the auxiliary problem: we establish a relationship between the sides of parallelogram and its diagonals.

We proceed from the fact that the diagonal cross-section parallelepiped - parallelogram.

For parallelogram true each of the following statements:

- The opposing sides are equal in pairs;
- Opposite angles are equal;
- Diagonals intersect and bisect the intersection point;
- The opposing sides are parallel;
- The sum of angles adjacent to the one side, is 180°
- Each diagonal divides the parallelogram into two congruent triangles.

Consider the following parallelogram ABCD.



We have $AC = d_1, DB = d_2, AD = BC = a, AB = DC = b, \angle DAB = \alpha$

Then $\angle ADC = 180° - \alpha$.

For triangle ΔDAB we note the law of cosines.

$$d_2^2 = a^2 + b^2 - 2ab \cdot \cos(\alpha)$$

Then we consider the triangle ΔADC and write for them the law of cosines.

$$d_1^2 = a^2 + b^2 - 2ab \cdot \cos(180° - \alpha) = a^2 + b^2 + 2ab \cdot \cos(\alpha)$$

Adding these equations, we obtain:

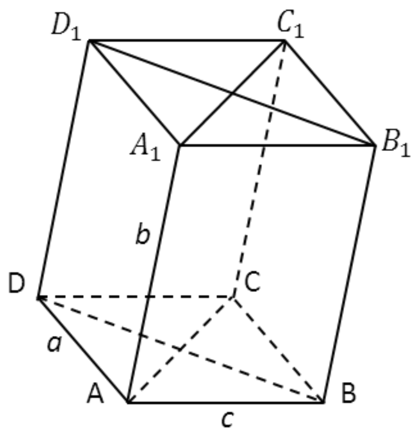
$$d_1^2 + d_2^2 = a^2 + b^2 + 2ab \cdot \cos(\alpha) + a^2 + b^2 - 2ab \cdot \cos(\alpha)$$

Simplify the obtained equation.

$$d_1^2 + d_2^2 = 2a^2 + 2b^2$$

Consider the parallelepiped $AA_1DD_1CC_1BB_1$ to prove statement from the condition of the given task.

Let the edges of the parallelepiped are a, b, c.



For the plane DD_1B_1B write the formula that we determined of the sum of the diagonals.

$$D_1B^2 + {DB_1}^2 = 2DB^2 + 2{BB_1}^2$$

Then we consider the plane AA_1C_1C write the formula that we determined of the sum of the diagonals.

$$A_1C^2 + AC_1{}^2 = 2AC^2 + 2AA_1{}^2$$

Now we add obtained earlier equalities:

$$D_1B^2 + DB_1{}^2 + A_1C^2 + AC_1{}^2 = 2DB^2 + 2BB_1{}^2 + 2AC^2 + 2AA_1{}^2$$

Now, to solve this equation we make the change of the sides.

Accordance with the terms of the problem, we noted the following.

$DB = d_2$ and $AC = d_1$

We have already identified that $d_1^2 + d_2^2 = 2a^2 + 2c^2$.

Now we can rewrite the equation.

$$D_1B^2 + DB_1{}^2 + A_1C^2 + AC_1{}^2 = 2DB^2 + 2BB_1{}^2 + 2AC^2 + 2AA_1{}^2$$
$$= 2(DB^2 + AC^2) + 2BB_1{}^2 + 2AA_1{}^2$$

According to the condition of the task we also know that $BB_1 = AA_1 = b$.

So, we can substitute into the equation.

$$D_1B^2 + DB_1{}^2 + A_1C^2 + AC_1{}^2 = 2(2a^2 + 2b^2) + 2b^2 + 2b^2$$

Simplify and combine like terms on the right side of the equation.

$$D_1B^2 + DB_1{}^2 + A_1C^2 + AC_1{}^2 = 2(2a^2 + 2c^2) + 2b^2 + 2b^2 = 4a^2 + 4c^2 + 4b^2$$

Thus we have the sum of squares of all the edges of the parallelepiped

Finally we can write the proof.

$D_1B^2 + DB_1{}^2 + A_1C^2 + AC_1{}^2 = 4a^2 + 4c^2 + 4b^2 = 4(a^2 + c^2 + b^2)$.