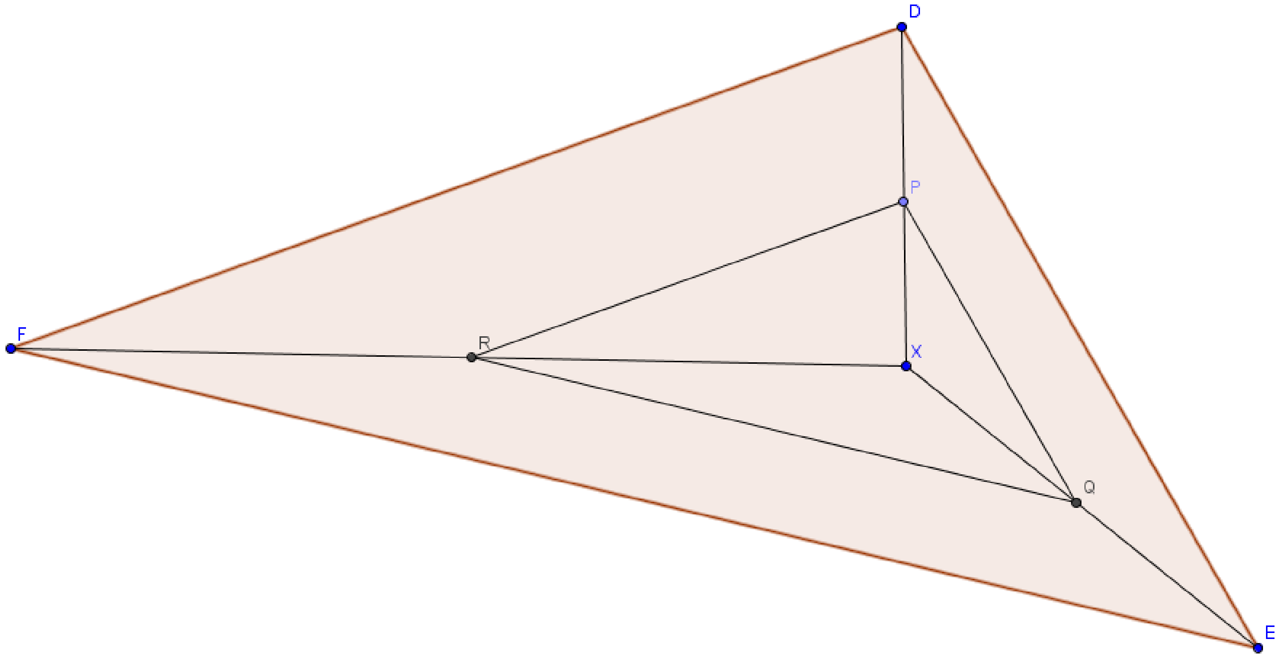
Answer on Question #43538 – Math - Analytic Geometry

Any point X inside triangle DEF is joined to its vertices. From a point P on DX, PQ is drawn parallel to DE, meeting XE at Q and QR is drawn parallel to EF, meeting XF in R. Prove that PR is parallel to DF.

Solution:



Let $\overrightarrow{XD} = \vec{a}$, $\overrightarrow{XE} = \vec{b}$, $\overrightarrow{XF} = \vec{c}$. Then $\overrightarrow{DE} = \vec{b} - \vec{a}$, $\overrightarrow{EF} = \vec{c} - \vec{b}$, $\overrightarrow{FD} = \vec{a} - \vec{c}$. Suppose that $\overrightarrow{XP} = \lambda\,\overrightarrow{XD} = \lambda\vec{a}$. The vector $\overrightarrow{PQ}$ is colliner to $\overrightarrow{DE}$, so $\overrightarrow{PQ} = \mu_1\overrightarrow{DE} = \mu_1(\vec{b} - \vec{a})$. Then $\overrightarrow{XQ} = \overrightarrow{XP} + \overrightarrow{PQ} = (\lambda - \mu_1)\vec{a} + \mu_1\vec{b}$. On the other hand $\overrightarrow{XQ} = \mu_2\overrightarrow{XE} = \mu_2\vec{b}$, so $(\mu_1 - \lambda)\vec{a} + \mu_1\vec{b} = \mu_2\vec{b}$. The vectors $\vec{a}$ and $\vec{b}$ are linear independent, so $\mu_1 = \lambda$, $\mu_2 = \mu_1 = \lambda$ and $\overrightarrow{XQ} = \lambda\vec{b}$. The vector $\overrightarrow{QR}$ is colliner to $\overrightarrow{EF}$, so $\overrightarrow{QR} = \mu_3\overrightarrow{EF} = \mu_3(\vec{c} - \vec{b})$. Then $\overrightarrow{XR} = \overrightarrow{XQ} + \overrightarrow{QR} = (\lambda - \mu_3)\vec{b} + \mu_3\vec{c}$. On the other hand $\overrightarrow{XR} = \mu_4\overrightarrow{XF} = \mu_4\vec{c}$, so $(\lambda - \mu_3)\vec{b} + \mu_3\vec{c} = \mu_4\vec{c}$. The vectors $\vec{b}$ and $\vec{c}$ are linear independent, so $\mu_3 = \lambda$, $\mu_4 = \mu_3 = \lambda$ and $\overrightarrow{XR} = \lambda\vec{c}$. Hence $\overrightarrow{RP} = \overrightarrow{XP} - \overrightarrow{XR} = \lambda(\vec{a} - \vec{c}) = \lambda\overrightarrow{FD}$, so vector $\overrightarrow{RP}$ is parallel to $\overrightarrow{FD}$ or $RP||FD$.