Answer on Question #43530 – Math - Statistics and Probability

The density function of a random variable X is given by

$$f(x) = \frac{1}{7\sqrt{2\pi}} e^{\frac{-(x+3.6)^2}{98}}.$$

Find its (a) math expectation, (b) variance and (c) distribution function.

Solution

We can write

$$f(x) = \frac{1}{7\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+3.6}{7}\right)^2}.$$

This is the density function of the general normal distribution with location parameter μ=-3.6 and scale parameter $\sigma = 7$.

After the transformation $x = \mu + z\sigma = -3.6 + 7z$ we can use the standard normal distribution density function:

$$f(x) = \frac{1}{7\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} = \frac{1}{7}\phi(z).$$

(a) math expectation

$$E(X) = E(\mu + Z\sigma) = \mu + \sigma E(Z)$$

$$E(Z) = \int_{-\infty}^{\infty} z\phi(z)dz = \int_{-\infty}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}\,dz = \int_{-\infty}^{0} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}\,dz + \int_{0}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}\,dz.$$

Using the simple substitution $u = \frac{1}{2}(z)^2$ we find

$$E(Z) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-u}\,du + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u}\,du = -\frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} = 0.$$

So, $E(X) = \mu = -3.6$.

(b) variance

$$Var(X) = Var(\mu + Z\sigma) = \sigma^2 Var(Z).$$

$$Var(Z) = E(z^2) = \int_{-\infty}^{\infty} z^2\phi(z)dz.$$

Integrate by parts, using the parts $u = z$ and $dv = z\phi(z)dz$. Thus $du = dz$ and $v = -\phi(z)$. Note that $z\phi(z) \to 0\ as\ z \to \infty\ and\ as\ z \to -\infty$. Thus, the integration by parts formula gives

$$Var(Z) = \int_{-\infty}^{\infty} \phi(z)dz = 1.$$

So, $Var(X) = \sigma^2 Var(Z) = \sigma^2 = 49$.

(c) distribution function

The normal distribution function $F(X)$ is

$$F(X) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{7\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t+3.6}{7}\right)^2} dt = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x+3.6}{7}\right),$$

where

$$\Phi(\mathrm{Z}) = \int_{-\infty}^{z} \phi(t)dt = \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}(t)^2}}{\sqrt{2\pi}} dt.$$