Answer on Question #42864 – Math - Linear Algebra

Let B1 ={(1,1),(1,2)} and B2 ={(1,0),(2,1)}. Find the matrix of the change of basis from B1 to B2

Solution

To find the matrix $P_{B_2 \leftarrow B_1}$ of the change of basis from $B_1 = \{e_1; e_2\} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ to $B_2 = \{u_1; u_2\} =$

$= \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$, we first express the basis vectors

$u_1 = \alpha_1 e_1 + \alpha_2 e_2$

$u_2 = \beta_1 e_1 + \beta_2 e_2$ or

$\begin{pmatrix}1\\0\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\2\end{pmatrix},$

$\begin{pmatrix}2\\1\end{pmatrix} = \beta_1 \begin{pmatrix}1\\1\end{pmatrix} + \beta_2 \begin{pmatrix}1\\2\end{pmatrix},$

It means

$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},$

$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}.$

Multiply equalities by the corresponding inverse matrix $\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}$ and get

$\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix},$

$\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}2\\1\end{pmatrix}$

Find matrix $\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}$ by row reduction:

$$\left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\1 & 2 & 0 & 1\end{array}\right) \sim \begin{array}{c}R_1 \leftarrow R_1\\R_2 \leftarrow R_2 - R_1\end{array} \sim \left(\begin{array}{cc|cc}1 & 1 & 1 & 0\\0 & 1 & -1 & 1\end{array}\right) \sim \begin{array}{c}R_1 \leftarrow R_1 - R_2\\R_2 \leftarrow R_2\end{array} \sim \left(\begin{array}{cc|cc}1 & 0 & 2 & -1\\0 & 1 & -1 & 1\end{array}\right).$$

So, $\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}.$

Finally

$$\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}.$$

Thus, $\begin{pmatrix}\alpha_1 & \alpha_2\\\beta_1 & \beta_2\end{pmatrix}^T = \begin{pmatrix}\alpha_1 & \beta_1\\\alpha_2 & \beta_2\end{pmatrix} = \begin{pmatrix}2 & 3\\-1 & -1\end{pmatrix}$ is the matrix of change of basis from B1 to B2