Answer on Question #42862 – Math - Linear Algebra

Problem. Use Gram-Schmidt orthogonalisation process to find an orthonormal basis for the subspace of $\mathbb{C}^4$ generated by the vectors $(1, i, 0, -i)$, $(-i, 0,1,2)$ and $(0, -i, 1,1)$.

Solution.

$a_1(1, i, 0, -i), a_2(-i, 0,1,2), a_3(0, -i, 1,1)$.

$b_1 = a_1 = (1, i, 0, -i)$.

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = (-i, 0,1,2) - (1, i, 0, -i) \cdot \frac{(-i) \cdot 1 + 0 \cdot (-i) + 1 \cdot 0 + 2 \cdot i}{1 \cdot 1 + i \cdot (-i) + 0 \cdot 0 + (-i) \cdot i} =$$

$$= (-i, 0,1,2) - (1, i, 0, -i) \cdot \frac{i}{3} = (-i, 0,1,2) - \left(\frac{i}{3}, \frac{-1}{3}, 0, \frac{1}{3}\right) = \frac{1}{3}(-4i, 1,3,5).$$

$$b_3 = a_3 - \frac{(a_3,b_2)}{(b_2,b_2)} b_2 - \frac{(a_3,b_1)}{(b_1,b_1)} b_1 =$$

$$= (0, -i, 1,1) - \frac{1}{3}(-4i, 1,3,5) \cdot \frac{\frac{1}{3}(0 \cdot 4i + (-i) \cdot 1 + 1 \cdot 3 + 1 \cdot 5)}{\frac{1}{9}((-4i) \cdot 4i + 1 \cdot 1 + 3 \cdot 3 + 5 \cdot 5)} - (1, i, 0, -i) \cdot \frac{0 \cdot 1 + (-i) \cdot (-i) + 1 \cdot 0 + 1 \cdot i}{1 \cdot 1 + i \cdot (-i) + 0 \cdot 0 + (-i) \cdot i} =$$

$$= (0, -i, 1,1) - (-4i, 1,3,5) \cdot \frac{8 - i}{51} - (1, i, 0, -i) \cdot \frac{i - 1}{3} =$$

$$= (0, -i, 1,1) - \frac{1}{51}(-32i - 4, 8 - i, 24 - 3i, 40 - 5i) - \frac{1}{3}(i - 1, -i - 1, 0, i + 1) =$$

$$= \frac{1}{51}(0 + 32i + 4 - 17i + 17, -51i - 8 + i + 17i + 17, 51 - 24 + 3i + 0, 51 - 40 + 5i - 17i - 17) =$$

$$= \frac{1}{51}(15i + 21, -33i + 9, 27 + 3i, -6 - 12i) = \frac{1}{17}(5i + 7, -11i + 3, 9 + i, -2 - 4i).$$

$b_1(1, i, 0, -i)$, $b_2\left(-\frac{4i}{3}, \frac{1}{3}, 1, \frac{5}{3}\right)$, $b_3\left(\frac{5i+7}{17}, \frac{-11i+3}{17}, \frac{9+i}{17}, \frac{-2-4i}{17}\right)$.

$|b_1|^2 = 1 \cdot 1 + i \cdot (-i) + 0 \cdot 0 + (-i) \cdot i = 3;\ |b_1| = \sqrt{3}$.

$|b_2|^2 = \left(-\frac{4i}{3}\right) \cdot \frac{4i}{3} + \frac{1}{3} \cdot \frac{1}{3} + 1 \cdot 1 + \frac{5}{3} \cdot \frac{5}{3} = \frac{51}{3^2};\ |b_2| = \frac{\sqrt{51}}{3}$.

$|b_3|^2 = \left(\frac{7+5i}{17}\right) \cdot \left(\frac{7-5i}{17}\right) + \left(\frac{3-11i}{17}\right) \cdot \left(\frac{3+11i}{17}\right) + \left(\frac{9+i}{17}\right) \cdot \left(\frac{9-i}{17}\right) + \left(\frac{-2-4i}{17}\right)\left(\frac{-2+4i}{17}\right) = \frac{306}{17^2};\ |b_3| = \frac{\sqrt{306}}{17}$.

Vectors b_1, b_2, b_3 form an orthogonal basis. We will normalize vectors b_1, b_2, b_3.

$c_1 = \frac{b_1}{|b_1|} = \frac{1}{\sqrt{3}}(1, i, 0, -i)$.

$c_2 = \frac{b_2}{|b_2|} = \frac{1}{\sqrt{51}}(-4i, 1,3,5)$.

$c_3 = \frac{b_3}{|b_3|} = \frac{1}{\sqrt{306}}(5i + 7, -11i + 3, 9 + i, -2 - 4i)$.

Answer: $\frac{1}{\sqrt{3}}(1, i, 0, -i), \frac{1}{\sqrt{51}}(-4i, 1,3,5), \frac{1}{\sqrt{306}}(5i + 7, -11i + 3, 9 + i, -2 - 4i)$.