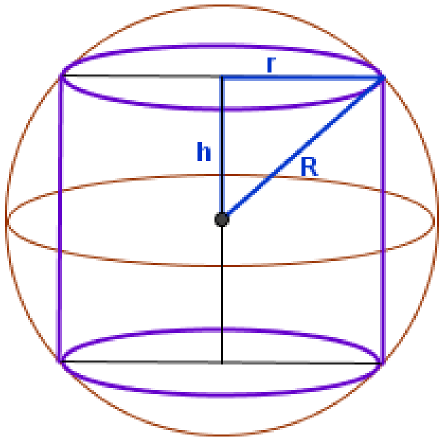
Answer on Question#42663 – Math - Calculus

Question:

What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere?

Solution:

Let R be the radius of sphere, r – radius of base of cylinder and h - **half** of cylinder's height.



It's known that volume of the sphere is

$$V_s = \frac{4}{3}\pi R^3$$

and volume of cylinder is

$$V_c = \pi r^2 \cdot 2h = 2\pi r^2 h$$

By Pythagorean theorem, $R^2 = h^2 + r^2$,

then $r^2 = R^2 - h^2$.

Hence $V_c = 2\pi(R^2 - h^2)h = 2\pi(R^2 h - h^3)$

To find max volume obtain derivative on h and equate it to zero.

$(V_c)' = 2\pi(R^2 - 3h^2)$

$2\pi(R^2 - 3h^2) = 0$

$R^2 - 3h^2 = 0$

$h^2 = \frac{R^2}{3}$

Then $h = \frac{R}{\sqrt{3}}$ and $r^2 = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$

If $h < \frac{R}{\sqrt{3}}$ then $R^2 - 3h^2 > 0$ and $(V_c)'$>0.

But if $h > \frac{R}{\sqrt{3}}$ then $R^2 - 3h^2 < 0$ and $(V_c)' < 0$.

It means that $h = \frac{R}{\sqrt{3}}$ is point of maximum of V_c and

$V_{c,max} = 2\pi \frac{2R^2}{3} \frac{R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$

Finally, fraction is

$$\frac{V_{c,max}}{V_s} = \frac{\frac{4\pi R^3}{3\sqrt{3}}}{\frac{4}{3}\pi R^3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577$$

Answer:

$$\frac{V_{c,max}}{V_s} = \frac{\sqrt{3}}{3} \approx 0.577$$