Answer on Question #42611-Math-Other

A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is between 0 and 1). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance H, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light.

Solution

r is a radius of a semicircle, horizontal side of the rectangle is equal $2r$, vertical side of the rectangle is equal $H-r$. Effective area of the window is

$$S = S_{\text{rectangular}} + kS_{\text{semicircular}} = 2r \cdot (H-r) + \frac{k\pi r^2}{2}.$$

$$\frac{dS}{dr} = 2H - 4r + k\pi r = 2H + (k\pi - 4)r.$$

$$\frac{dS}{dr} = 0 \to 2H + (k\pi - 4)r = 0 \to r = \frac{2H}{4 - k\pi}.$$

The ratio is

$$\frac{(H-r)}{2r} = \frac{1}{2}\left(\frac{H}{r} - 1\right) = \frac{1}{2}\left(\frac{H(4-k\pi)}{2H} - 1\right) = \frac{2-k\pi}{4}.$$

If $k \le \frac{2}{\pi}$ this solution is valid, but if $k \ge \frac{2}{\pi}$ the ratio is negative.

If $k \ge \frac{2}{\pi}$ the function $S = 2r \cdot (H-r) + \frac{k\pi r^2}{2}$ $(r \le H)$ has maximum value at $r = H$:

$$S = 2H \cdot (H-H) + \frac{k\pi H^2}{2} = 0 + \frac{k\pi H^2}{2} = \frac{k\pi H^2}{2}.$$

This means that the window should be semicircular with no rectangular part.

If $k \ge \frac{2}{\pi}$, the ratio is zero:

$$\frac{(H-H)}{2H} = 0.$$

Answer: If $\boldsymbol{k \le \frac{2}{\pi}}$ the ratio is $\boldsymbol{\frac{2-k\pi}{4}}$; if $\boldsymbol{k \ge \frac{2}{\pi}}$, the ratio is zero.