Answer on Question #42602– Math – Differential Calculus|Equation

Question:

Using laplace transforms solve the following ordinary differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2(t^2 + t + 1)\ ; y(0) = 2\ , y'(0) = 0$$

Solution:

Let Y(s)=L[y(t)](s). Instead of solving directly for y(t), we derive a new equation for Y(s). Once we find Y(s), we inverse transform to determine y(t).

The first step is to take the Laplace transform of both sides of the original differential equation. We have

$$L[y'' + 3y' + 2y](s) = L[2(t^2 + t + 1)\ ](s)$$

If we look at the left-hand side, we have

$$L[y'' + 3y' + 2y](s) = L[y''](s) + 3L[y'](s) + 2L[y](s)$$

Now use the formulas for the L[y''], L[y'']and L[y']:

$$L[y'] = sL[y] - y(0) = sY(s) - 1;$$

$$L[y''] = s^2L[y] - sy(0) - y'(0) = s^2Y(s) - s - 2;$$

Hence, we have

$$L[y'' + 3y' + 2y](s) = s^2Y(s) - s - 2 + 3(sY(s) - 1) + 2Y(s).$$

If we look at the right-hand side, we have

$$L[2(t^2 + t + 1)](s) = 2L[t^2](s) + 2L[t](s) + 2L[1](s)$$

Now use the formulas for the $L[t^2]$, L[t]and L[1]:

$$L[t^2] = \frac{2}{s^3}, \qquad L[t] = \frac{1}{s^2}, \quad L[1] = \frac{1}{s}.$$

The Laplace-transformed differential equation is

$$s^2Y(s) - s - 2 + 3(sY(s) - 1) + 2Y(s) = \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s}$$

$$(s^2 + 3s + 2)Y(s) = s + 5 + \frac{4}{s^3} + \frac{2}{s^2} + \frac{2}{s}$$

Or

$$(s^2+3s+2)Y(s)=\frac{(s+1)(s^3+4\,s^2-2\,s+4)}{s^3}$$

This is a linear algebraic equation for Y(s)! We have converted a differential equation into a algebraic equation! Solving for Y(s), we have

$$Y(s)=\frac{(s+1)(s^3+4\,s^2-2\,s+4)}{s^3(s^2+3s+2)}$$

We can simplify this expression using the method of partial fractions:

$$Y(s)=\frac{2}{s^3}-\frac{2}{s^2}-\frac{2}{s+2}+\frac{3}{s}\quad (for\ s\neq -1)$$

Recall the inverse transforms:

$$L^{-1}\left[\frac{1}{s^3}\right](t)=\frac{1}{2t^2},\quad L^{-1}\left[\frac{1}{s^2}\right](t)=t,\ \ L^{-1}\left[\frac{1}{s+2}\right](t)=e^{-2t},\quad L^{-1}\left[\frac{1}{s}\right](t)=1$$

Using linearity of the inverse transform, we have

$$y(t)=L^{-1}\left[\frac{2}{s^3}-\frac{2}{s^2}-\frac{2}{s+2}+\frac{3}{s}\right]=2*\frac{1}{2}t^2-2*t-e^{-2t}+3*1=$$

$$=t^2-2\,t-\ e^{-2\,t}+3;$$

Answer. $y(t)=t^2-2\,t-\ e^{-2\,t}+3;$