Answer on Question #42377 – Math – Integral Calculus:

Check the continuity of the function:

$$f(x,y) = \begin{cases} \dfrac{7x^2y}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}.$$

Solution.

We need to check the continuity of f at every point (x_0, y_0).

1) $(x_0, y_0) \neq (0,0)$:

$$\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = \lim_{\substack{x\to x_0 \\ y\to y_0}} \frac{7x^2y}{x^2+y^2} = \frac{7x_0^2y_0}{x_0^2+y_0^2} = f(x_0,y_0);$$

So, f is continuous at every point $(x_0, y_0) \neq (0,0)$.

2) $(x_0, y_0) = (0,0)$:

$$\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{7x^2y}{x^2+y^2} = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{7y}{1+\left(\frac{y}{x}\right)^2};$$

$$\lim_{\substack{x\to 0 \\ y\to 0}} \left|\frac{7y}{1+\left(\frac{y}{x}\right)^2}\right| \leq \lim_{\substack{x\to 0 \\ y\to 0}} \left|\frac{7y}{1}\right| = 7\lim_{\substack{x\to 0 \\ y\to 0}} |y| = 0 \Rightarrow \lim_{\substack{x\to 0 \\ y\to 0}} \frac{7y}{1+\left(\frac{y}{x}\right)^2} = 0;$$

Hence:

$$\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) = 0 = f(0,0);$$

So, f is continuous at $(0,0)$.

We conclude that f is continuous in $\mathbb{R} \times \mathbb{R}$.