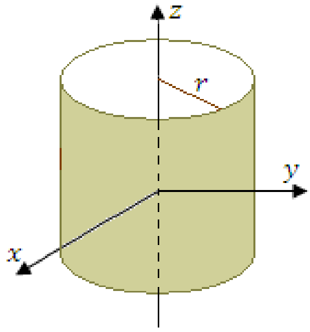
Answer to Question #42376 - Math - Integral Calculus

Question: Evaluate the integral of $f(x,y,z) = x + 2y - z$ over the cylinder bounded by $x^2 + y^2 = 4$ and $z = 1$.

Solution. For this solution, it is assumed that the cylinder is bounded below by $z = 0$.



It is most convenient to calculate the integral

$$\iiint_{\substack{x^2+y^2\le 4\\ 0\le z\le 1}} (x + 2y - z)dx\,dy\,dz$$

using the cylindrical coordinates r, θ, z:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

In terms of cylindrical coordinates, $dV = dx\,dy\,dz = r\,d\theta\,dr\,dz$.

Furthermore, the cylinder $\{(x,y,z)\colon\ x^2 + y^2 \le 4, 0 \le z \le 1\}$ becomes $\{(r,\theta,z)\colon 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1\}$, and the function to integrate is $f(r,\theta,z) = r(\cos\theta + 2\sin\theta) - z$.

Thus, we have

$$\iiint_{\substack{x^2+y^2\le 4\\ 0\le z\le 1}} (x + 2y - z)dx\,dy\,dz =$$

$$= \int_0^1\int_0^{2\pi}\int_0^2 (r(\cos\theta + 2\sin\theta) - z)r\,d\theta\,dr\,dz = \int_0^{2\pi}\int_0^2 (r^2(\cos\theta + 2\sin\theta) - r\frac{z^2}{2})\Big|_{z=0}^{z=1} d\theta\,dr = \int_0^2\int_0^{2\pi} (r^2(\cos\theta + 2\sin\theta) - \frac{r}{2})\,d\theta\,dr =$$

$$= \int_0^2\int_0^{2\pi} (r^2(\cos\theta + 2\sin\theta) - \frac{r}{2})\,d\theta\,dr =$$

$$= \int_0^{2\pi} \left(\frac{r^3}{3}(\cos\theta + 2\sin\theta) - \frac{r^2}{4}\right)\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} \left(\frac{8}{3}(\cos\theta + 2\sin\theta) - \frac{4}{4}\right)d\theta$$

$$= \left(\frac{8}{3}\sin\theta - \frac{16}{3}\cos\theta - \theta\right)\Big|_{\theta=0}^{\theta=2\pi} = -\frac{16}{3} - 2\pi + \frac{16}{3} = -2\pi.$$

Answer. -2π.