Answer on question 42307 – Math – Linear Algebra

Let

$$A = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}$$

a) Find the adjoint of A. Find the inverse of A from the adjoint of A.

b) Find the characteristic and minimal polynomials of A. Hence find its eigenvalues and eigenvectors.

c) Why is A diagonalisable? Find a matrix P such that P^(−1) AP is diagonal.

d) Verify Cayley-Hamilton theorem for A. Hence, find the inverse of A.

Solution

a) Adjoint matrix is the matrix formed by taking the transpose of the cofactor matrix of a given original matrix. To find it we need to find the following determinants

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 7 & -6 \\ 12 & -11 \end{vmatrix} = -77 + 72 = -5;$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 6 & -6 \\ 12 & -11 \end{vmatrix} = 66 - 72 = -6;$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 6 & 7 \\ 12 & 12 \end{vmatrix} = 72 - 84 = -12;$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & -4 \\ 12 & -11 \end{vmatrix} = 44 - 48 = -4;$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & -4 \\ 12 & -11 \end{vmatrix} = -55 + 48 = -7;$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 4 \\ 12 & 12 \end{vmatrix} = -60 + 48 = -12;$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & -4 \\ 7 & -6 \end{vmatrix} = -24 + 28 = 4;$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 5 & -4 \\ 6 & -6 \end{vmatrix} = 30 - 24 = 6;$$

$$A_{12} = (-1)^{3+3}\begin{vmatrix} 5 & 4 \\ 6 & 7 \end{vmatrix} = 35 - 24 = 11;$$

Therefore

$$adj\ A = \begin{pmatrix} -5 & -4 & 4 \\ -6 & -7 & 6 \\ -12 & -12 & 11 \end{pmatrix}$$

Using the triangle rule we find the determinant of A

$$\det(A) = \begin{vmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{vmatrix} = 5 * -7 * (-11) + 6 * 12 * (-4) + 4 * 12 * (-6) -$$
$$-(12 * 7 * (-4) + 6 * 4 * (-11) + 12 * (-6) * 5) = -1.$$

Using the formula

$$A^{-1} = \frac{1}{\det(A)} * adj\ A$$

We obtain

$$A^{-1} = -1\begin{pmatrix} -5 & -4 & 4 \\ -6 & -7 & 6 \\ -12 & -12 & 11 \end{pmatrix} = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}.$$

b) The characteristic polynomial of the matrix is

$$P(x) = \det(xI - A) = \begin{vmatrix} x-5 & -4 & 4 \\ -6 & x-7 & 6 \\ -12 & -12 & x+11 \end{vmatrix} = (x-5)(x-7)(x+11) +$$
$$+288 + 288 + 48(x-7) - 24(x+11) + 72(x-5) =$$
$$= x^3 - x^2 - x + 1 = (x-1)^2(x+1)$$

Thus the distinct eigenvalues of A are -1 and 1. Since m_A divides $P(x)$ and every eigenvalue of A is a root of m_A, we must have that $m_A(x) = x^2 - 1$ or $m_A(x) = (x-1)^2(x+1)$. To check which of these works, we start with the one of the smallest degree:

$$A^2 - I = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}\begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 25+24-48 & 20+28-48 & -20-24+44 \\ 30+42-72 & 24+49-72 & -24-42+66 \\ 60+72-132 & 48+84-132 & -48-72+121 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the minimal polynomial of A is $m_A(x) = x^2 - 1$.

As we said before the eigenvalues of A are -1 and 1 with algebraic multiplicity is 2.

Let us find the eigenvectors

1) X=-1:

$$\begin{pmatrix} 6 & 4 & -4 \\ 6 & 8 & -6 \\ 12 & 12 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

2) X=1:

$$\begin{pmatrix} 4 & 4 & -4 \\ 6 & 6 & -6 \\ 12 & 12 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_2 - c_1 \\ c_1 \\ c_2 \end{pmatrix}$$

Where c_1 and c_2 are constants which are not equal to zero at the same time. So we get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

c) Use the Theorem: An n x n matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Check whether the eigenvectors of A are linearly independent. For it find the determinant of the matrix consisted with these vectors

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = -3 + 6 - 2 = 1$$

Therefore the eigenvectors of A are linearly independent and the matrix A is diagonalizable.

P is the matrix that consists of the eigenvectors. Check it

$$P^{-1}AP = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} -1 & -1 & 1 \\ 6 & 6 & -5 \\ 3 & 4 & -3 \end{pmatrix}\begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ 6 & 6 & -5 \\ 3 & 4 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \\ 6 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d) The Cayley–Hamilton theorem states that "substituting" the matrix A for λ in this polynomial results in the zero matrix

$$P(A) = A^3 - A^2 - A + 1 = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}^3 - \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix}^2 -$$

$$- \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} -$$

$$- \begin{pmatrix} 5 & 4 & -4 \\ 6 & 7 & -6 \\ 12 & 12 & -11 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the Theorem holds.

As we know

$$AA^{-1} = I$$

As we could notice $A^2 = I$ therefrom $A = A^{-1}$.