Answer on Question #42306 – Math - Linear Algebra

Use the properties of determinants to evaluate the following determinant:

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Solution

Let

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}.$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get,

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

here, take common $(a+b+c)$ from C_1 & C_2.

Now, applying $R_3 \to R_3 - (R_1 + R_2)$, we get,

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} = \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix},$$

applying $C_1 \to aC_1$ & $C_2 \to bC_2$

$$\Delta = \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix},$$

applying $C_1 \to C_1 + C_3$ & $C_2 \to C_2 + C_3$

$$\Delta = \frac{(a+b+c)^2}{ab} ab \cdot 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix},$$

here, take a, b, c common from R_1, R_2 & R_3

$$\Delta = 2ab(a+b+c)^2 \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix},$$

expand along R_3

$$\Delta = 2ab(a+b+c)^2[(b+c)(c+a) - ab] = 2ab(a+b+c)^2[ab + ac + bc + c^2 - ab]$$
$$= 2abc(a+b+c)^2(a+b+c) = 2abc(a+b+c)^3.$$

Answer: $\mathbf{2abc(a+b+c)^3}$.