Answer on Question #42303 – Math – Linear Algebra:

a) Let a quadratic form have the expression $x^2 + y^2 + 2z^2 + 2xy + 3xz$ with respect to the standard basis $B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$. Find its expression with respect to the basis $B_2 = \{(1,1,1),(0,1,0),(0,1,1)\}$.

b) Consider the quadratic form $Q(x,y,z) = 2x^2 - 4xy + y^2 + 4xz + 3z^2$.
i) Find a symmetric matrix A such that $Q = X^T AX$.
ii) Find the orthogonal canonical reduction of the quadratic form.
iii) Find the principal axes of the form.
iv) Find the rank and signature of the form.

Solution.

a)

$$\begin{cases} u = x + y + z \\ v = y \\ w = y + z \end{cases} \Rightarrow \begin{cases} x = u - y - z \\ y = v \\ z = w - y \end{cases} \Rightarrow \begin{cases} x = u - w \\ y = v \\ z = w - v \end{cases};$$

$$Q(x,y,z) = x^2 + y^2 + 2z^2 + 2xy + 3xz =$$
$$= (u-w)^2 + v^2 + 2(w-v)^2 + 2v(u-w) + 3(u-w)(w-v) =$$
$$= u^2 - 2uw + w^2 + v^2 + 2(w^2 - 2vw + v^2) + 2(uv - vw) +$$
$$+3(uw - uv + vw - w^2) = u^2 + 3v^2 - uv + uw - 3vw.$$

Hence:

$$Q(u,v,w) = u^2 + 3v^2 - uv + uw - 3vw.$$

b)

i) $$Q(x,y,z) = 2x^2 - 4xy + y^2 + 4xz + 3z^2 = 2\left(x^2 + 2x(z-y)\right) + y^2 + 3z^2 =$$
$$= 2(x+z-y)^2 - 2(z-y)^2 + y^2 + 3z^2 = 2(x+z-y)^2 - y^2 + z^2 + 4yz =$$
$$= 2(x-y+z)^2 + (z+2y)^2 - 4y^2 - y^2 = 2(x-y+z)^2 - 5y^2 + (2y+z)^2;$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x-y+z \\ y \\ 2y+z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

$$Q = \begin{pmatrix} u \\ v \\ w \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 2 & 0 & 0 \\ -2 & -5 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \begin{pmatrix} 2 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

Hence:

$$A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$

ii)

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x-y+z \\ y \\ 2y+z \end{pmatrix} \Rightarrow Q(u,v,w) = 2(x-y+z)^2 - 5y^2 + (2y+z)^2 =$$

$$= 2u^2 - 5v^2 + w^2.$$

iii)

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x-y+z \\ y \\ 2y+z \end{pmatrix} \Rightarrow \begin{cases} u = (1,-1,1) \\ v = (0,1,0) \\ w = (0,2,1) \end{cases} - \text{principal axes.}$$

iv) l

$$rank(Q) = rank(A) = rank\begin{pmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3;$$

$$Q(u,v,w) = 2u^2 - 5v^2 + w^2 \Rightarrow sign(Q) = 2 - 1 = 1.$$