Answer on Question #42285-Math-Real Analysis

Let S⊆R be non-empty. Prove that if a number u in R has the properties (i) for every n∈N the number u-1/n is not an upper bound of S, and (ii) for every number n∈N the number u + 1/n is upper bound S, then

u=sup S.

Solution

1) Let's prove the first part.

Assume that there is some $n \in N$ such that $\left(u - \frac{1}{n}\right)$ is an upper bound of S.

By the definition supremum is the lowest upper bound. But easy to see that $\left(u - \frac{1}{n}\right) < u$ and thus u isn't the lowest bound.

So it's not true that there is some $n \in N$ such that $\left(u - \frac{1}{n}\right)$ is an upper bound of S.

It means that for every number $n \in N$ the number $\left(u - \frac{1}{n}\right)$ is not an upper bound of S.

2) Let's prove the second part.

For any $x \in S$ it's true that $x \le u$, because $u = \sup S$.

At the same time for any $n \in N$ it's true that $u < u + \frac{1}{n}$.

So we have for any $x \in S$: $x \le u < u + \frac{1}{n}$. From the transitivity of real numbers $x < u + \frac{1}{n}$.

Thus for any $n \in N$ $\left(u + \frac{1}{n}\right)$ is greater than any element of S, which means that $\left(u + \frac{1}{n}\right)$ is an upper bound.