Answer on Question #42210 – Math – Topology

Question. Suppose that $A \subset B \subset X$ and A is dense in closure of B and B is dense in X. Then A is also dense in X.

Proof. Recall that a subset $A \subset X$ is *dense* if for every open $U \in X$ the intersection $A \cap U \neq \varnothing$.

Also recall that a *closure*, $\overline{A}$, of A in X is the set of all $x \in X$ such that for every open $V \subset X$ containing x the intersection $A \cap V \neq \varnothing$.

First we establish the following lemma:

Lemma. *A is dense in X if and only if $\overline{A} = X$.*

Proof. Suppose A is dense in X. Let $x \in X$ and $V \subset X$ be any open subset containing x. Then $A \cap V \neq \varnothing$ as A is dense in X, and so $x \in \overline{A}$, that is $\overline{A} = X$.

Conversely, suppose $\overline{A} = X$. Let $U \subset X$ be any open non-empty set and $x \in U$. As $x \in \overline{A} = X$, we have that $U \cap A \neq \varnothing$, and so A is dense in X. □

Now the proof is easy. Due to lemma the assumption that A is dense in closure of B, means that $\overline{A} = \overline{B}$, while the assumption that B is dense in X means that $\overline{B} = X$. Thus

$$\overline{A} = \overline{B} = X,$$

and so A is dense in X.