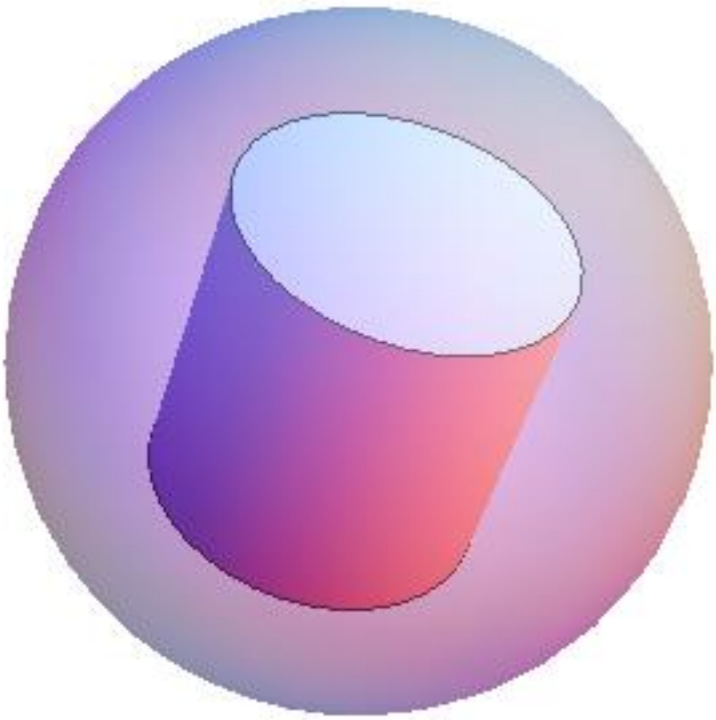
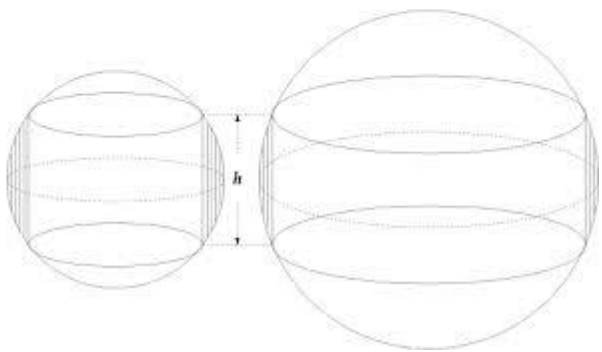
Answer on Question #42042 – Math – Multivariable Calculus

Determine the volume of the solid bound between the sphere $x^2+y^2+z^2=R^2$ and the cylinder $x^2+y^2=r^2$, where $R>r$ (such shapes are called spherical rings or napkin rings).





Solution.

The body is inside the cylinder but bounded by the sphere outside.
Let introduce cylindrical coordinates:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}.$$

$dx\,dy\,dz = \rho\,d\rho\,d\varphi\,dz$. $V: \begin{cases} \rho^2 + z^2 = R^2 \\ \rho = r \end{cases}$.

Range for variables: $\varphi \in [0; 2\pi]$, $\rho \in [0; r]$, $z \in \left[-\sqrt{R^2 - \rho^2};\ \sqrt{R^2 - \rho^2}\right]$.

The volume of the solid: $V = \iiint_V dxdydz = \int_0^{2\pi} d\varphi \int_0^r \rho\,d\rho \int_{-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} dz = \varphi\Big|_0^{2\pi} \cdot \int_0^r \rho\,d\rho \cdot z\Big|_{-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} =$

$$= 2\pi \cdot \int_0^r \rho\,d\rho \cdot 2\sqrt{R^2 - \rho^2} = -2\pi \cdot \int_0^r \sqrt{R^2 - \rho^2}\,d\left(R^2 - \rho^2\right) = -2\pi \cdot \left.\frac{\left(R^2 - \rho^2\right)^{3/2}}{\frac{3}{2}}\right|_0^r =$$

$$= \frac{4\pi}{3}\left[R^3 - \left(R^2 - r^2\right)^{3/2}\right].$$

Answer: $\frac{4\pi}{3}\left[R^3 - \left(R^2 - r^2\right)^{3/2}\right]$.