Answer on Question #42009 - Math - Vector Calculus

Vector $\vec{r}$ has following components in Descartes coordinate system: $\vec{r}=(x;y;z)$, $r=\sqrt{x^2+y^2+z^2}$.

Let us evaluate this cross product using the Leibniz rule for nabla operator:

$$[\nabla,\frac{\vec{r}}{r}]=\frac{1}{r}[\nabla,\vec{r}]+[\nabla(\frac{1}{r}),\vec{r}]$$

Let us use the property of nabla operator $\nabla f(r)=\frac{\partial f(r)}{\partial r}\cdot\frac{\vec{r}}{r}$. It is easy to prove it. Let us take x-component of nabla operator: $\frac{\partial}{\partial x}f(\sqrt{(x^2+y^2+z^2)})=\frac{\partial f(r)}{\partial\sqrt{(x^2+y^2+z^2)}}\cdot\frac{2x}{2\sqrt{(x^2+y^2+z^2)}}=\frac{\partial f(r)}{\partial r}\cdot\frac{x}{r}$. The same result takes place for y and z derivative (with y and z in numerator respectively).

Thus, $\nabla(\frac{1}{r})=\frac{-1}{r^2}\cdot\frac{\vec{r}}{r}=\frac{-\vec{r}}{r^3}$, so $[\nabla(\frac{1}{r}),\vec{r}]=\frac{-1}{r^3}[\vec{r},\vec{r}]=0$ (according to the properties of cross product, $[\vec{a},\vec{a}]=0$).

The term $[\nabla,\vec{r}]\equiv rot\, r$ is equal to zero. By definition, $[\nabla,\vec{r}]_k=\varepsilon_{ijk}\nabla_i(\vec{r})_j$. Since Levi-Civita tensor is not equal to zero only if its indexes are different, and $\nabla_i\vec{r}_j=\delta_{ij}$, all components of $[\nabla,\vec{r}]$ are equal to zero, thus $[\nabla,\vec{r}]=0$.

Finally, $[\nabla,\frac{\vec{r}}{r}]=\frac{1}{r}[\nabla,\vec{r}]+[\nabla(\frac{1}{r}),\vec{r}]=0$.