Answer on Question #41715 – Math – Other

Question.

Solve the Volterra equation $\varphi(t) + \int (t-x)\varphi(x)dx = t$, for $0 \le t \le 1$

Using the successive approximation. Note that the answer should be $\varphi(t) = sin(t)$

Solution.

Consider general solution method of Volterra integral equation:

$$\varphi(t) - \lambda \int_a^t K(t,x)\varphi(x)dx = g(t)$$

$$a \le t \le b$$

We find a solution in the form of a series:

$$\varphi(t) = \varphi_0(t) + \varphi_1(t) \cdot \lambda + \varphi_2(t) \cdot \lambda^2 + \cdots$$

$$\varphi_0(t) = g(t);\ \varphi_n(t) = \int_a^t K(t,x)\varphi_{n-1}(x)dx\,;\ n = 1,2,3\ldots$$

Now, let come back to our case:

$$\varphi(t) - \int (x-t)\varphi(x)dx = t$$

$$0 \le t \le 1$$

So,

$$g(t) = t; K(t,x) = (x-t); \lambda = 1; a = 0; b = 1$$

Therefore,

$$\varphi_0(t) = g(t) = t$$

$$\varphi_1(t) = \int_a^t K(t,x)\varphi_0(x)dx = \int_0^t (x-t)xdx = \int_0^t (x^2 - tx)dx = \frac{t^3}{3} - \frac{t^3}{2} = -\frac{t^3}{6} = -\frac{t^3}{3!}$$

$$\varphi_2(t) = \int_a^t K(t,x)\varphi_1(x)dx = \int_0^t -(x-t)\frac{x^3}{3!}dx = \int_0^t \frac{tx^3}{3!} - \frac{x^4}{3!}dx = \frac{t^5}{3!}\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{t^5}{120} = \frac{t^5}{5!}$$

$$\varphi_3(t) = \int_a^t K(t,x)\varphi_2(x)dx = \int_0^t (x-t)\frac{x^5}{5!}xdx = \int_0^t \frac{x^6}{5!} - \frac{tx^5}{5!}xdx = \frac{t^7}{5!}\left(\frac{1}{7} - \frac{1}{6}\right) =$$
$$= -\frac{t^7}{5!\cdot 6\cdot 7} = -\frac{t^7}{7!}$$

So,

$$\varphi(t) = \varphi_0(t) + \varphi_1(t)\cdot\lambda + \varphi_2(t)\cdot\lambda^2 + \varphi_3(t)\cdot\lambda^3 + \cdots = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sin(t)$$

Maclaurin series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

So,

$$\varphi(t) = \sin(t)$$

Answer.

$\varphi(t) = \sin(t)$