Answer on Question # 41714 – Math – Real Analysis

Assume a = a(t) ∈ L1(0,1)∩C(0,∞), ∞ ≥ a(0+) > a(∞) ≥ 0 and a(t) ≥ 0, a′(t) ≤ 0, a′′(t) ≥ 0, a′′′(t) ≤ 0, on (0,∞). Let h(t) = ta(t) on (0,∞). Then one of the following is true. There exists a number ε > 0 such that h is increasing on (0, ε). Or, there is no such interval. Which is correct? Prove it.

Solution.

$h(t) \in L_1(0,1) \cap C(0;\infty)$, as a composition of $a(t)$ and continuous function t. $h(t)$ has the same order of derivatives as $a(t)$.

Consider the derivative of $h(t)$:

$$h'(t) = \left(t \cdot a(t)\right)' = a(t) + ta'(t).$$

We obtain that $ta'(t) \leq 0$, as $t > 0$ and $a'(t) \leq 0\ on\ (0;\infty)$.

So, $h(t)$ is increasing on interval $(0,\varepsilon)$ if $a(t) + ta'(t) > 0$ on it. As function $a(t)$ is decreasing and t is increasing on $(0;\infty)$ there exist $t \in (0,\varepsilon) \subset (0;\infty):\ a(t) > ta'(t)$.

For example, consider $a(t) = e^{-t}\ on\ (0,\infty)$. $a(t) \in L_1(0,1) \cap C^{\infty}(0,\infty)$. And

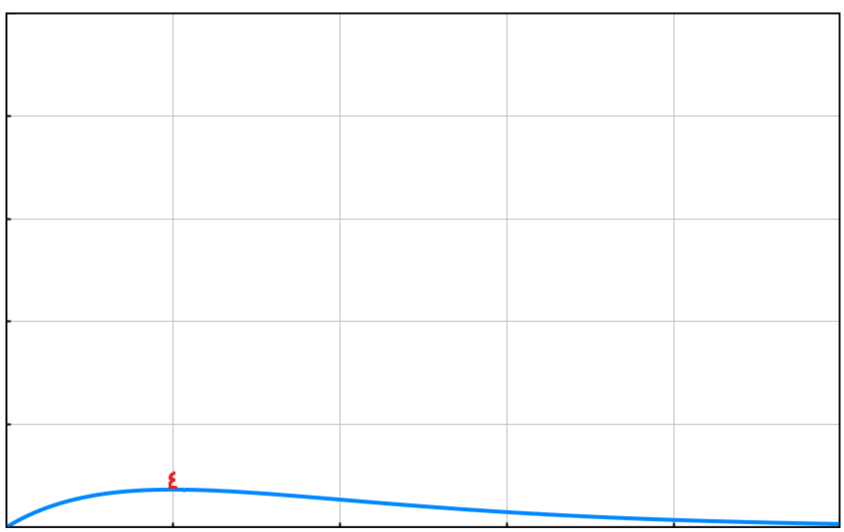
$$a'(t) = -e^{-t} \leq 0,$$

$$a''(t) = e^{-t} \geq 0, a'''(t) = -e^{-t} \leq 0\ on\ (0,\infty).$$

$$h(t) = te^{-t}\ on\ (0;\infty).$$

$$h'(t) = e^{-t} - te^{-t} > 0\ for\ t \in (0,\varepsilon), \qquad here\ \varepsilon = 1.$$

$$h(t) = te^{-t}:$$



Answer. There exists a number $\varepsilon \geq 0$ such that h is increasing on $(0,\varepsilon)$.