Answer on question 41325 – Math – Functional Analysis

Let X be the set of all real-valued functions x on the interval [0,1], and let x≦y mean that x(t) ≦y(t) for all t∈[0,1]. Show that this defines a partial ordering. Is it a total ordering? Does X have maximal elements?

Solution

Recall the definition of partial ordering.

A partial order is a binary relation "≤" over a set *P* which is reflexive, antisymmetric, and transitive, i.e., which satisfies for all *a*, *b*, and *c* in *P*:

- *a* ≤ *a* (reflexivity);
- if *a* ≤ *b* and *b* ≤ *a* then *a* = *b* (antisymmetry);
- if *a* ≤ *b* and *b* ≤ *c* then *a* ≤ *c* (transitivity).

So we need to check whether this conditions are satisfying.

1) $x(t) \leq x(t)$ for all $t \in [0;1]$;
2) If $x(t) \leq y(t)$ and $y(t) \leq x(t)$ for all $t \in [0;1]$ then $x(t) = y(t)$ for all $t \in [0;1]$ (from the definition of equal functions);
3) If $x(t) \leq y(t)$ and $y(t) \leq z(t)$ for all $t \in [0;1]$ it is obviously that $x(t) \leq z(t)$ for all $t \in [0;1]$;

Hence $\leq$ defines a partial ordering.

If in addition the trichotomy law satisfies than it defines total order. (For any $a, b \in S$, either $a \leq b$ or $b \leq a$.)

4) For any two functions $x(t)$ and $y(t)$ either $x(t) \leq y(t)$ or $y(t) \leq x(t)$ at each point t. But for all $t \in [0;1]$ it is not true (at some point the first inequality can satisfies and at another the second).

So, this is not the total order.

As the functions can take any real value, including infinity, than the set X don’t have the maximal element.