Answer on Question # 40904 – Math – Functional Analysis

Question. If Z is an $(n-1)$-dimensional subspace of an n-dimensioned vector space X, show that Z is the null space of a suitable linear functional f on X, which is uniquely determined to within a scalar multiple.

Solution. Let $e_1, \dots, e_{n-1}$ be a basis for Z. Since $\dim X = n$, there exists a vector $e_n \in X$ such that the vectors

$$e_1,\ \dots,\ e_{n-1},\ e_n$$

constitute a basis for X.

Then each $x \in X$ can be uniquely represented as a linear combination of $\{e_i\}_{i=1}^n$, that is

$$x = a_1 e_1 + \cdots + a_n e_n$$

for a unique n-tuple of numbers $(a_1, \dots, a_n)$ and these numbers are called the *coordinates* of x in this basis. Also notice that $x = (a_1, \dots, a_n) \in Z$ if and only if $a_n = 0$.

Furthermore, every linear functional $f : X \to \mathbb{R}$ is uniquely determined by its values on basis vectors $e_1, \dots, e_n$. Indeed, denote $a_i = f(e_i)$, then for any $x = (x_1, \dots, x_n) = x_1 e_1 + \cdots + x_n e_n$,

$$f(x) = f(x_1, \dots, x_n) = f(x_1 e_1 + \cdots + x_n e_n) = x_1 f(e_1) + \cdots + x_n f(e_n) = a_1 x_1 + \cdots + a_n x_n.$$

We should construct a linear functional $f : X \to \mathbb{R}$ such that $f(x) = 0$ if and only if $x \in Z$. Since $e_1, \dots, e_{n-1} \in Z$ and $e_n \notin Z$, it follows that

$$f(e_1) = \cdots = f(e_{n-1}) = 0$$

and

$$f(e_n) \neq 0.$$

Therefore

$$f(x) = f(x_1 e_1 + \cdots + x_n e_n) = x_n f(e_n),$$

so f is uniquely determined by its *non-zero* value of e_n.

Thus for any $a \in \mathbb{R} \setminus \{0\}$ the functional $f_a(x_1, \dots, x_n) = a x_n$ has the required properties: $f(x) = 0$ if and only if $x \in Z$. In particular, such f exists.

Moreover, if $f_b(x_1, \dots, x_n) = b x_n$ is another such linear functional with $b \in \mathbb{R} \setminus \{0\}$, then

$$f_b(x) = b x_n = \tfrac{b}{a} \cdot a x_n = \tfrac{b}{a} \cdot f_a(x),$$

and so they differs by constant multiple $\frac{b}{a}$.

Thus a linear functional f on X for which Z is a null space exist and is uniquely determined to within a scalar multiple.