Answer on Question #40829 – Math – Functional Analysis

Question. Let Z be a proper subspace of an n-dimensional vector space X, and let f be a linear functional on Z. Show that f can be extended linearly to X, that is, there is a linear functional F on X such that $F|_Z = f$.

Proof. Suppose $\dim Z = k < n$, and let $e_1, \dots, e_k$ be a basis for Z, that is a maximal collection of linearly independent vectors in Z. It is known that every basis of a subspace of a finite-dimensional vector space X can be extended to a basis of all of X. So let us extend the above basis of Z to a basis

$$e_1, \dots, e_k,\ e_{k+1}, \dots, e_n$$

of all of X. Then every $x \in X$ can be uniquely represented as a linear combination of $\{e_i\}_{i=1}^n$, that is

$$x = a_1e_1 + \cdots + a_ke_k + a_{k+1}e_{k+1} + \cdots + a_ne_n.$$

for a unique n-tuple of numbers $(a_1, \dots, a_n)$ being the coordinates of x in this basis. Also notice that $x = (a_1, \dots, a_n) \in Z$ if and only if $a_{k+1} = \cdots = a_n = 0$.

Using coordinates in the above basis define a function $F : X \to \mathbb{R}$ by the following formula:

$$F(a_1, \dots, a_k, a_{k+1}, \dots, a_n) = f(a_1, \dots, a_k),$$

in other owrds

$$F(a_1e_1 + \cdots + a_ke_k + a_{k+1}e_{k+1} + \cdots + a_ne_n) = F(a_1e_1 + \cdots + a_ke_k).$$

We claim that F is a linear functional such that $F|_Z = f$.

Indeed, let $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n) \in X$, and $s, t \in \mathbb{R}$. Then

$$\begin{aligned} F(sx + ty) &= F(sa_1 + tb_1, \dots, sa_n + tb_n) \\ &= f(sa_1 + tb_1, \dots, sa_k + tb_k) \\ &= sf(a_1, \dots, a_k) + tf(b_1, \dots, b_k) \\ &= sF(a_1, \dots, a_n) + tF(b_1, \dots, b_n) \\ &= sF(x) + tF(y), \end{aligned}$$

so F is a linear functional.

Moreover, if

$$x = a_1e_1 + \cdots + a_ke_k = (a_1, \dots, a_k, \underbrace{0, \dots, 0}_{n-k}) \in Z$$

then

$$F(x) = F(a_1e_1 + \cdots + a_ke_k) = f(a_1e_1 + \cdots + a_ke_k) = f(x),$$

so $F|_Z = f(x)$.