Answer on Question #40755, Math, Vector Calculus

If the force field defined by vector F=(3x^2yz-3y)i^ + (x^3z - 3x)j^ + (x^3y+2z)k^ conservative? if so, find the scalar potential associated with the vector F.

Solution

$$\vec{F} = (3x^2yz - 3y)\vec{\imath} + (x^3z - 3x)\vec{\jmath} + (x^3y + 2z)\vec{k} = M(x,y,z)\vec{\imath} + N(x,y,z)\vec{\jmath} + P(x,y,z)\vec{k}$$

Then, $\vec{F}$ is conservative if and only if

$$\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}, \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}.$$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(x^3y + 2z) = 3x^2y.$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^3y + 2z) = x^3.$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3x^2yz - 3y) = 3x^2z - 3.$$

$$\frac{\partial M}{\partial z} = \frac{\partial}{\partial z}(3x^2yz - 3y) = 3x^2y.$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^3z - 3x) = 3x^2z - 3.$$

$$\frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(x^3z - 3x) = x^3.$$

So $\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} = 3x^2y, \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 3x^2z - 3, \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} = x^3$ and <u>the force field $\vec{\mathrm{F}}$ is conservative.</u>

Let's find the scalar potential f associated with the vector $\vec{F}$

$$\frac{\partial f}{\partial x} = M = (3x^2yz - 3y),$$

$$\frac{\partial f}{\partial y} = N = (x^3z - 3x),$$

$$\frac{\partial f}{\partial z} = P = (x^3y + 2z).$$

If we integrate the first of the three equations with respect to x, we find that

$$f(x,y,z) = \int (3x^2yz - 3y)dx = x^3yz - 3yx + g(y,z).$$

where $g(y,z)$ is a constant dependent on y and z variables. We then calculate the partial derivate with respect to y from this equation and match it with the equation of above.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3yz - 3yx + g(y,z)\right) = x^3z - 3x + \frac{\partial g}{\partial y} = (x^3z - 3x).$$

This means that the partial derivative of g with respect to y is 0, thus eliminating y from g entirely and leaving at as a function of z alone.

$$f(x, y, z) = x^3yz - 3yx + h(z).$$

We then repeat the process with the partial derivative with respect to z

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\big(x^3yz - 3yx + h(z)\big) = x^3y + \frac{dh}{dz} = (x^3y + 2z)$$

which means that

$$\frac{dh}{dz} = (2z)$$

so we can find $h(z)$ by integrating:

$$h(z) = z^2 + c.$$

Therefore,

$$f(x, y, z) = x^3yz - 3yx + z^2 + c.$$