Answer on Question #40608 – Math - Calculus

Find the Taylor's Polynomial of degree n about x = c and the remainder for the function f given

1) f(x)=1/3x+4 and c=0

2) f(x)= sin x and c= π/6

Solution:

The Taylor's Polynomial of degree n about x=c is the power series

$$T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n =$$

$$= \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i.$$

1. In this case we have

$$f(x) = \frac{1}{3x+4} = (3x+4)^{-1} \Rightarrow f(0) = \frac{1}{4};$$

$$f'(x) = -1 \cdot 3(3x+4)^{-2} \Rightarrow f'(0) = \frac{-1 \cdot 3}{4^2};$$

$$f''(x) = -1 \cdot (-2) \cdot 3^2(3x+4)^{-3} \Rightarrow f''(0) = \frac{-1 \cdot (-2) \cdot 3^2}{4^3};$$

…

$$f^{(n)}(x) = -1 \cdot (-2) \cdot \ldots \cdot (-n) \cdot 3^n(3x+4)^{-(n+1)} \Rightarrow$$

$$\Rightarrow f^{(n)}(0) = \frac{-1 \cdot (-2) \cdot \ldots \cdot (-n) \cdot 3^n}{4^{n+1}} = \frac{(-1)^n}{4} \cdot \left(\frac{3}{4}\right)^n \cdot n!.$$

So

$$T_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{4} \cdot \left(\frac{3}{4}\right)^i \cdot i! \cdot \frac{1}{i!}(x-0)^i = \sum_{i=0}^{n} \frac{(-1)^i}{4} \cdot \left(\frac{3}{4}\right)^i \cdot x^i.$$

2. In this case we have

$$f(x) = \sin x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{1}{2};$$

$$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2};$$

$$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{6}\right) = -\frac{1}{2};$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2};$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2};$$

$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Thus

$$f^{(2i)}(x) = (-1)^i \sin x \Rightarrow f^{(2i)}\left(\frac{\pi}{6}\right) = \frac{(-1)^i}{2};$$

$$f^{(2i+1)}(x) = (-1)^i \cos x \Rightarrow f^{(2i+1)}\left(\frac{\pi}{6}\right) = \frac{(-1)^i\sqrt{3}}{2}.$$

So if $n = 2k$ then

$$T_n(x) = T_{2k}(x) = \sum_{i=0}^{k} \frac{(-1)^i}{2 \cdot (2i)!}\left(x - \frac{\pi}{6}\right)^{2i} + \sum_{i=0}^{k-1} \frac{(-1)^i\sqrt{3}}{2 \cdot (2i+1)!}\left(x - \frac{\pi}{6}\right)^{2i+1};$$

If $n = (2k+1)$ then

$$T_n(x) = T_{2k+1}(x) = \sum_{i=0}^{k} \frac{(-1)^i}{2 \cdot (2i)!}\left(x - \frac{\pi}{6}\right)^{2i} + \sum_{i=0}^{k} \frac{(-1)^i\sqrt{3}}{2 \cdot (2i+1)!}\left(x - \frac{\pi}{6}\right)^{2i+1}.$$

Answer:

1.

$$T_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{4} \cdot \left(\frac{3}{4}\right)^i \cdot x^i$$

2.

$$T_n(x) = T_{2k}(x) = \sum_{i=0}^{k} \frac{(-1)^i}{2 \cdot (2i)!}\left(x - \frac{\pi}{6}\right)^{2i} + \sum_{i=0}^{k-1} \frac{(-1)^i\sqrt{3}}{2 \cdot (2i+1)!}\left(x - \frac{\pi}{6}\right)^{2i+1};$$

or

$$T_n(x) = T_{2k+1}(x) = \sum_{i=0}^{k} \frac{(-1)^i}{2 \cdot (2i)!}\left(x - \frac{\pi}{6}\right)^{2i} + \sum_{i=0}^{k} \frac{(-1)^i\sqrt{3}}{2 \cdot (2i+1)!}\left(x - \frac{\pi}{6}\right)^{2i+1}.$$