Answer on Question #40492 – Math - Differential Calculus

Using the method of undetermined coefficients, write the trial solution of the equation y"+2y'+5y=x(e)1/x cos 2x and hence solve it .

Solution.

We have the differential equation

$$y'' + 2y' + 5y = xe^{-x}\cos 2x$$

The general solution will be the sum of the complementary solution and particular (trial) solution.

Find the complementary solution by solving

$$y'' + 2y' + 5y = 0$$

Assume a solution will be proportional to $e^{\lambda x}$ for some constant λ. Substitute $y = e^{\lambda x}$ into the differential equation:

$$\left(e^{\lambda x}\right)'' + 2\left(e^{\lambda x}\right)' + 5e^{\lambda x} = 0$$

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 5e^{\lambda x} = 0$$

Factor out $e^{\lambda x}$:

$$(\lambda^2 + 2\lambda + 5)e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$ for any finite λ, the zeros must come from the polynomial:

$$\lambda^2 + 2\lambda + 5 = 0$$

Solve for λ:

$$\lambda = -1 + 2i \text{ or } \lambda = -1 - 2i$$

The roots $\lambda = -1 \pm 2i$ give $y_1 = c_1 e^{(-1+2i)x}, y_2 = c_2 e^{(-1-2i)x}$ as solutions, where c_1 and c_2 are arbitrary constants.

The general solution is the sum of the above solutions:

$$y = y_1 + y_2 = \frac{c_1}{e^{(1-2i)x}} + \frac{c_2}{e^{(1+2i)x}}$$

Apply Euler's identity $e^{\alpha+i\beta} = e^{\alpha}\cos\beta + ie^{\alpha}\sin\beta$:

$$y = c_1\left(\frac{\cos 2x}{e^x} + i\frac{\sin 2x}{e^x}\right) + c_2\left(\frac{\cos 2x}{e^x} - i\frac{\sin 2x}{e^x}\right)$$

Regroup:

$$y = \frac{(c_1 + c_2)\cos 2x}{e^x} + \frac{i(c_1 - c_2)\sin 2x}{e^x}$$

Redefine $c_1 + c_2$ as c_1 and $i(c_1 - c_2)$ as c_2, since these are arbitrary constants:

$$y=\frac{c_1\cos 2x}{e^x}+\frac{c_2\sin 2x}{e^x}$$

Determine the particular solution to

$$y''+2y'+5y=xe^{-x}\cos 2x$$

by the method of undetermined coefficients. The particular solution is of the form:

$$y_p=x\left(\frac{A\cos 2x}{e^x}+\frac{Bx\cos 2x}{e^x}+\frac{C\sin 2x}{e^x}+\frac{Dx\sin 2x}{e^x}\right),$$

where $\frac{A\cos 2x}{e^x}+\frac{Bx\cos 2x}{e^x}+\frac{C\sin 2x}{e^x}+\frac{Dx\sin 2x}{e^x}$ was multiplied by x to account for $\frac{\cos 2x}{e^x}$ in the complementary solution.

Solve for unknown constants A,B,C,D.

Compute y_p':

$$y_p'=\left(\frac{Ax\cos 2x}{e^x}+\frac{Bx^2\cos 2x}{e^x}+\frac{Cx\sin 2x}{e^x}+\frac{Dx^2\sin 2x}{e^x}\right)'=\frac{A\cos 2x}{e^x}-\frac{Ax\cos 2x}{e^x}-$$

$$-\frac{2Ax\sin 2x}{e^x}-\frac{Bx^2\cos 2x}{e^x}+\frac{2Bx\cos 2x}{e^x}-\frac{2Bx^2\sin 2x}{e^x}+\frac{2Cx\cos 2x}{e^x}+\frac{C\sin 2x}{e^x}-\frac{Cx\sin 2x}{e^x}+$$

$$+\frac{2Dx^2\cos 2x}{e^x}-\frac{Dx^2\sin 2x}{e^x}+\frac{2Dx\sin 2x}{e^x}$$

Compute y_p'':

$$y_p''=\left(\frac{Ax\cos 2x}{e^x}+\frac{Bx^2\cos 2x}{e^x}+\frac{Cx\sin 2x}{e^x}+\frac{Dx^2\sin 2x}{e^x}\right)''=-\frac{4Ax\cos 2x}{e^x}+$$

$$+A\cos 2x\left(-\frac{2}{e^x}+\frac{x}{e^x}\right)-4A\left(e^{-x}-\frac{x}{e^x}\right)\sin 2x-\frac{4Bx^2\cos 2x}{e^x}+B\cos 2x\left(\frac{2}{e^x}+\frac{x^2}{e^x}-\frac{4x}{e^x}\right)-$$

$$-4B\left(-\frac{x^2}{e^x}+\frac{2x}{e^x}\right)\sin 2x+4C\cos 2x\left(e^{-x}-\frac{x}{e^x}\right)-\frac{4Cx\sin 2x}{e^x}+C\left(-\frac{2}{e^x}+\frac{x}{e^x}\right)\sin 2x+$$

$$+4D\cos 2x\left(-\frac{x^2}{e^x}+\frac{2x}{e^x}\right)-\frac{4Dx^2\sin 2x}{e^x}+D\left(\frac{2}{e^x}+\frac{x^2}{e^x}-\frac{4x}{e^x}\right)\sin 2x$$

Substitute the particular solution y_p into the differential equation and simplify:

$$\frac{(2B+4C)\cos 2x}{e^x}+\frac{8Dx\cos 2x}{e^x}+\frac{(-4A+2D)\sin 2x}{e^x}-\frac{8Bx\sin 2x}{e^x}=\frac{x\cos 2x}{e^x}$$

Equate the coefficients of $e^{-x}\cos 2x$ on both sides of the equation:

$$2B+4C=0$$

And similarly:

$$8D=1$$

$$-4A+2D=0$$

$$-8B = 0$$

Solve the system:

$$A = \frac{1}{16}$$

$$B = 0$$

$$C = 0$$

$$D = \frac{1}{8}$$

Substitute A, B, C, D into y_p:

$$y_p = \frac{x \cos 2x}{16e^x} + \frac{x^2 \sin 2x}{8e^x}$$

So the general solution is:

$$y = \frac{c_1 \cos 2x}{e^x} + \frac{c_2 \sin 2x}{e^x} + \frac{x \cos 2x}{16e^x} + \frac{x^2 \sin 2x}{8e^x}$$

Answer:

$$y(x) = \frac{c_1 \cos 2x}{e^x} + \frac{c_2 \sin 2x}{e^x} + \frac{x \cos 2x}{16e^x} + \frac{x^2 \sin 2x}{8e^x}$$