Answer on Question #40483, Math, Differential Calculus | Equations

The ends A and B of a rod 20cm long have the temperature at 30 degree centigrade and 80 degree centigrade respectively until steady state prevails. The temperatures of the ends are changed to 40 degree centigrade to 60 degree centigrade respectively. Find the temperature distribution in the rod at time t.

Solution

At time $t = 0$ steady state prevails:

$\frac{\partial^2 u(x,0)}{\partial x^2} = 0$, for $0 \le x \le 20$,

$u(0,0) = 30$ and $u(20,0) = 80$.

The solution to this boundary value problem is easily found, since the general solution of the differential equation is $u(x) = Ax + B$; where A and B are arbitrary constants. Then the boundary conditions reduce to $u(0) = B = 30$ and $u(20) = A \cdot 20 + B = 80$.

We conclude that $B = 30$ and $A = \frac{80-30}{20} = \frac{5}{2}$.

So the steady-state initial temperature is

$$u(x,0) = \frac{5}{2}x + 30.$$

We will start this section by solving the initial/boundary value problem
$u_t(x,t) = ku_{xx}(x,t)$ for $t > 0\ and\ 0 < x < 20$,

$u(0,t) = 40\ and\ u(20,t) = 60$, for $t > 0$,

$u(x,0) = \frac{5}{2}x + 30$, for $0 \le x \le 20$.

It is useful for both mathematical and physical purposes to split the problem into two parts. We first find the steady-state temperature that satisfies the boundary conditions. A steady-state temperature is one that does not depend on time. Then $u_t = 0$, so the heat equation simplifies to $u_{xx} = 0$. : Hence we are looking for a function $u_s(x)$ defined for $0 \le x \le 20$ such that
$\frac{\partial^2 u_s(x)}{\partial x^2} = 0$, for $0 < x < 20$,

$u_s(0,t) = 40$ and $u_s(20,t) = 60$ for $t > 0$.

The solution to this boundary value problem is easily found, since the general solution of the differential equation is $u_s(x) = Ax + B$; where A and B are arbitrary constants. Then the boundary conditions reduce to
$u_s(0) = B = 40$ and $u_s(20) = A \cdot 20 + B = 60$.

We conclude that $B = 40$ and $A = \frac{60-40}{20} = 1$.

So the steady-state temperature is

$$u_S(x,0) = x + 40.$$

It remains to find $v = u - u_s$. It will be a solution to the heat equation, since both u and u_s are, and the heat equation is linear. The boundary and initial conditions that v satisfies can be calculated from those for u and u_s. Thus, $v = u - u_s$, must satisfy

$v_t(x,t) = kv_{xx}(x,t)$ for $t > 0$ and $0 < x < 20$,

$u(0,t) = 0$ and $u(20,t) = 0$, for $t > 0$,

$v(x,0) = u(x,0) - u_s = \frac{5}{2}x + 30 - (x + 40) = \frac{3}{2}x - 10$, for $0 \le x \le 20$.

Having found the steady-state temperature us and the temperature v, the solution to the original problem is $u(x,t) = v(x,t) + u_s(x)$.

Let's hunt for solutions in the product form
$v(x,t) = X(x)T(t)$,

where $T(t)$ is a function of t and $X(x)$ is a function of x.

When we insert $v(x,t) = X(x)T(t)$ into the heat equation $v_t(x,t) = kv_{xx}(x,t)$, we get

$$X(x)T'(t) = kX"(x)T(t).$$

The key step is to separate the variables by bringing everything depending on t to the left, and everything depending on x to the right. Dividing equation by $kX(x)T(t)$, we get

$$\frac{T'(t)}{kT(t)} = \frac{X"(x)}{X(x)}.$$

Since x and t are independent variables, the only way that the left-hand side, a function of t, can equal the right-hand side, a function of x, is if both functions are constant. Consequently, there is a constant that we will write as $-\lambda$; such that $\frac{T'(t)}{kT(t)} = -\lambda$ and $\frac{X"(x)}{X(x)} = -\lambda$ or

$T'(t) - \lambda kT(t) = 0$ and $X"(x) + \lambda X(x) = 0$.

The first equation has the general solution

$$T(t) = Ce^{-\lambda kt}.$$

Suppose that $\lambda > 0$ and $\lambda = \omega^2$. Then the differential equation is $X"(x) + \omega^2 X(x) = 0$, which has the general solution

$$X(x) = a\cos\omega x + b\sin\omega x.$$

For this solution the boundary condition $X(0) = 0$ becomes $a = 0$. Then the boundary condition $X\ (20) = 0$ becomes

$$b\sin 20\omega = 0.$$

We are only interested in nonzero solutions, so we must have $\sin 20\omega = 0$. This occurs if $20\omega = n\pi$ for some positive integer n. When this is true we have the eigenvalue $\lambda = \omega^2 = \frac{n^2\pi^2}{400}$. For any nonzero constant b, $X(x) = b\sin\left(\frac{n\pi x}{20}\right)$ is an eigenfunction.

Finally, we get the product solutions,

$$v_n(x,t) = e^{-\frac{n^2\pi^2 kt}{400}}\sin\left(\frac{n\pi x}{20}\right),$$

for n = 1, 2, 3, … .

We are naturally led to consider the infinite series

$$v(x,t) = \sum_{n=1}^{\infty} b_n\, v_n(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{n^2\pi^2 kt}{400}} \sin\left(\frac{n\pi x}{20}\right).$$

Using the series definition for v, the initial condition becomes

$$v(x,0) = \frac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{20}\right),$$

where

$$b_n = \frac{2}{20}\int_0^{20} \left(\frac{3}{2}x - 10\right) \sin\left(\frac{n\pi x}{20}\right) dx = \frac{3}{20}\int_0^{20} x \sin\left(\frac{n\pi x}{20}\right) dx - \int_0^{20} \sin\left(\frac{n\pi x}{20}\right) dx. = +\frac{20}{\pi n}(1-(-1)^n).$$

$$b_n = -\frac{3}{20}\frac{400}{\pi n}(-1)^n + \frac{20}{\pi n}(1-(-1)^n) = \frac{20}{\pi n}(1-4\cdot(-1)^n).$$

$$v(x,t) = \sum_{n=1}^{\infty} \frac{20}{\pi n}(1-4\cdot(-1)^n) \sin\left(\frac{n\pi x}{20}\right).$$

The temperature distribution in the rod at time t is

$$u(x,t) = x + 40 + \sum_{n=1}^{\infty} \frac{20}{\pi n}(1-4\cdot(-1)^n) \sin\left(\frac{n\pi x}{20}\right).$$