Answer on question 40471 – Math – Differential equation

Solve the differential equation

$$p^2 + 2py\cot x - y^2 = 0,$$

where $p = y'$.

Solution:

We have

$$(y')^2 + 2y'y\cot x - y^2 = 0,$$

$$\frac{(y')^2 + 2y'y\cot x - y^2}{y^2} = 0, \qquad y \neq 0,$$

$$\left(\frac{y'}{y}\right)^2 + 2\frac{y'}{y}\cot x - 1 = 0.$$

Denote

$$z = \frac{y'}{y}.$$

Thus we have next equation

$$z^2 + 2z\cot x - 1 = 0,$$

$$D = (2\cot x)^2 - 4\cdot 1\cdot(-1) = 4\cot^2 x + 4 = 4(1+\cot^2 x) = \frac{4}{\sin^2 x} > 0.$$

So we get two cases

1)

$$z = \frac{-2\cot x + \sqrt{\frac{4}{\sin^2 x}}}{2} = -\frac{\cos x}{\sin x} + \frac{1}{\sin x} = \frac{1-\cos x}{\sin x},$$

$$\frac{y'}{y} = \frac{1-\cos x}{\sin x},$$

$$\frac{dy}{y} = \frac{1-\cos x}{\sin x}dx,$$

$$\int \frac{dy}{y} = \int \frac{1-\cos x}{\sin x}dx,$$

$$\ln y = \int \frac{1-\cos x}{\sin x}dx.$$

We have

$$\int \frac{1-\cos x}{\sin x}dx = -\int \frac{1-\cos x}{\sin^2 x}d(\cos x) = -\int \frac{1-\cos x}{1-\cos^2 x}d(\cos x) =$$

$$= -\int \frac{1-\cos x}{(1-\cos x)(1+\cos x)}d(\cos x) = -\int \frac{1}{1+\cos x}d(\cos x) =$$

$$= -\int \frac{1}{1+\cos x}d(1+\cos x) = -\ln(1+\cos x) + C.$$

So we get

$$\ln y = -\ln(1+\cos x) + C,$$

$$\ln y = \ln\frac{1}{1+\cos x} + C,$$

$$y(x) = \frac{c}{1+\cos x},$$

where $c = e^C$.

2)

$$z = \frac{-2\cot x - \sqrt{\frac{4}{\sin^2 x}}}{2} = -\frac{\cos x}{\sin x} - \frac{1}{\sin x} = -\frac{1+\cos x}{\sin x},$$

$$\frac{y'}{y} = -\frac{1+\cos x}{\sin x},$$

$$\frac{dy}{y} = -\frac{1+\cos x}{\sin x}dx,$$

$$\int \frac{dy}{y} = -\int \frac{1+\cos x}{\sin x}dx,$$

$$\ln y = -\int \frac{1+\cos x}{\sin x}dx.$$

We have

$$-\int \frac{1+\cos x}{\sin x}dx = \int \frac{1+\cos x}{\sin^2 x}d(\cos x) = \int \frac{1+\cos x}{1-\cos^2 x}d(\cos x) =$$

$$= \int \frac{1+\cos x}{(1-\cos x)(1+\cos x)}d(\cos x) = -\int \frac{1}{1-\cos x}d(1-\cos x) =$$

$$= -\int \frac{1}{1-\cos x}d(1-\cos x) = -\ln(1-\cos x) + C.$$

So we get

$$\ln y = -\ln(1 - \cos x) + C,$$

$$\ln y = \ln \frac{1}{1 - \cos x} + C,$$

$$y(x) = \frac{c}{1 - \cos x},$$

where $c = e^C$.

Answer:

$$y(x) = \frac{c}{1 + \cos x}$$

or

$$y(x) = \frac{c}{1 - \cos x}.$$