Answer on Question #40469, Math, Differential Calculus | Equations

Find the complete integral of $p^2 + q^2 - 2px - 2qy + 1 = 0$.

Solution.

$$f(x, y, z, p, q) = p^2 + q^2 - 2px - 2qy + 1 = 0$$

Charpit's auxiliary equation are:

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{p(p - x) + q(q - y)} = \frac{dx}{p - x} = \frac{dy}{q - y}$$

Taking first two fractions ,

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating , $\ln p = \ln q + \ln a$ or $p = aq$.

Putting $p = aq$ into $f(x, y, z, p, q)$, we get:

$$a^2 q^2 + q^2 - 2aqx - 2qy + 1 = 0$$

$$q^2(a^2 + 1) - 2q(ax + y) + 1 = 0$$

Solve for q

$$q = \frac{2(ax + y) \pm \sqrt{4(ax + y)^2 - 4(a^2 + 1)}}{2(a^2 + 1)}$$

Let's $(ax + y) = u$ and $(a^2 + 1) = b = const$

$$q = \frac{u \pm \sqrt{u^2 - b}}{b}$$

$$p = aq = \frac{a}{b}(u \pm \sqrt{u^2 - b})$$

We know that

$$dz = pdx + qdy = aqdx + qdy = q(adx + dy) = qd(ax + y) = qdu$$

Integrating

$$z = \int \frac{u \pm \sqrt{u^2 - b}}{b} du = \frac{u^2}{2b} \pm \frac{1}{b} \int \sqrt{u^2 - b} du =$$

$$= \frac{u^2}{2b} \pm \frac{1}{2} u\sqrt{u^2 - b} - \frac{1}{2} b \ln\left(\sqrt{u^2 - b} + u\right) + C =$$

$$= \frac{(ax + y)^2}{2b} \pm \frac{1}{2}(ax + y)\sqrt{(ax + y)^2 - b} - \frac{1}{2} b \ln\left(\sqrt{(ax + y)^2 - b} + (ax + y)\right) + C$$

Answer: $z = \frac{(ax+y)^2}{2b} \pm \frac{1}{2}(ax + y)\sqrt{(ax + y)^2 - b} - \frac{1}{2} b \ln\left(\sqrt{(ax + y)^2 - b} + (ax + y)\right) + C$