Answer on Question #40385, Math, Differential Calculus

Task was solve the differential equation:
$(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$ (1)
Solution:
Let's find the integrating factor $\mu(y)$ by which you multiply the equation, that it was resolved in the total derivatives. For this let $R(x,y) = xy^2 - x^2$, $S(x,y) = 3x^2y^2 + x^2y - 2x^3 + y^2$.

Now we have equation like $\mu(y)R(x.y) + \mu(y)S(x,y)\frac{dy}{dx} = 0$

Condition for the existence the integrating factor $\mu(y)$ is $\frac{\partial}{\partial y}\left(\mu(y)R(x.y)\right) = \frac{\partial}{\partial}\left(\mu(y)S(x,y)\right)$.

Substitute $R(x,y)$ and $S(x,y)$ into (1):

$$\frac{d\mu(y)}{dy}\left(xy^2 - x^2\right) + 2yx\mu(y) = \mu(y)\left(6xy^2 + 2xy - 6x^2\right)$$

Collect the common multipliers

$$\frac{d\mu(y)}{dy}\left(xy^2 - x^2\right) = \mu(y)\left(6xy^2 - 6x^2\right)$$

$$\frac{d\mu(y)}{\mu(y)} = \frac{\left(6xy^2 - 6x^2\right)}{\left(xy^2 - x^2\right)}dy = 6dy.$$

Integrate to y: $\ln\mu(y) = 6y \Rightarrow \mu(y) = e^{6y}$.
Now we found integrating factor $\mu(y)$. Multiply our equation for $\mu(y)$:
$e^{6y}(xy^2 - x^2)dx + e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$
Let $P(x,y) = e^{6y}(xy^2 - x^2)$, $Q(x,y) = e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)$

This is an exact equation, because $\frac{\partial P(x,y)}{\partial y} = 6xe^{6y}(y^2 - x) + 2xe^{6y}y = \frac{\partial Q(x,y)}{\partial x}$

Define $f(x,y)$ such that $\frac{\partial f(x,y)}{\partial x} = P(x,y)$, $\frac{\partial f(x,y)}{\partial y} = Q(x,y)$.

Then solution will be given by $f(x,y) = C$, where C is arbitrary constant.

Integrate $\frac{\partial f(x,y)}{\partial x}$ with respect to x in order to find $f(x,y)$:

$f(x,y) = \int e^{6y}(xy^2 - x^2)dx = e^{6y}\left(\frac{y^2x^2}{2} - \frac{x^3}{3}\right) + g(y)$, where $g(y)$ is an arbitrary function of y.

Differentiate $f(x,y)$ with respect to y in order to find $g(y)$:

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(e^{6y}\left(\frac{y^2x^2}{2} - \frac{x^3}{3}\right) + g(y)\right) = e^{6y}yx^2 + 6e^{6y}\left(\frac{y^2x^2}{2} - \frac{x^3}{3}\right) + \frac{dg(y)}{dy}$$

Substitute into $\frac{\partial f(x,y)}{\partial y} = Q(x,y)$:

$$e^{6y}yx^2 + 6e^{6y}\left(\frac{y^2x^2}{2} - \frac{x^3}{3}\right) + \frac{dg(y)}{dy} = e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)$$

$$e^{6y}yx^2 + e^{6y}\left(3y^2x^2 - 2x^3\right) + \frac{dg(y)}{dy} = e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)$$

$$\frac{dg(y)}{dy} = e^{6y}y^2$$

Integrate $\frac{dg(y)}{dy}$ with respect to y:

$$g(y)=\int e^{6y}y^2dy=\frac{1}{6}\int y^2de^{6y}=\frac{1}{6}y^2e^{6y}-\frac{1}{6}\int e^{6y}\cdot 2ydy=\frac{1}{6}y^2e^{6y}-\frac{1}{3}\left(\frac{1}{6}ye^{6y}-\frac{1}{6}\int e^{6y}dy\right)=$$

$$=\frac{1}{6}y^2e^{6y}-\frac{1}{3}\left(\frac{1}{6}ye^{6y}-\frac{1}{36}e^{6y}\right)=\frac{1}{108}e^{6y}\left(18y^2-6y+1\right)$$

Substitute $g(y)$ into $f(x,y)$:

$$f(x,y)=\int e^{6y}(xy^2-x^2)dx=e^{6y}\left(\frac{y^2x^2}{2}-\frac{x^3}{3}\right)+\frac{1}{108}e^{6y}\left(18y^2-6y+1\right)$$

The solution is $f(x,y)=C$

$$e^{6y}\left(\frac{y^2x^2}{2}-\frac{x^3}{3}\right)+\frac{1}{108}e^{6y}\left(18y^2-6y+1\right)=C$$

Answer:

$$e^{6y}\left(\frac{y^2x^2}{2}-\frac{x^3}{3}\right)+\frac{1}{108}e^{6y}\left(18y^2-6y+1\right)=C$$