

Answer on Question#40158, Math, Linear Algebra

Find the rank of the quadratic form $2x^2 + 2y^2 + 3z^2 + xy + 4yz + 5xz$.

Solution.

Find a canonical expression of the quadratic form $(x, y, z) = 2x^2 + 2y^2 + 3z^2 + xy + 4yz + 5xz$, by using Lagrange's Reduction.

$$\begin{aligned} f(x, y, z) &= \left(2x^2 + \frac{1}{8}y^2 + \frac{25}{8}z^2 - xy + 5xz - \frac{5}{4}yz - \frac{1}{8}y^2 - \frac{25}{8}z^2 + \frac{5}{4}yz \right) + 2y^2 + 3z^2 \\ &\quad + 4yz = \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \right)^2 + \frac{15}{8}y^2 - \frac{1}{8}z^2 + \frac{25}{4}yz. \end{aligned}$$

The quadratic form $\frac{15}{8}y^2 - \frac{1}{8}z^2 + \frac{25}{4}yz$ is in two variables y and z and does not depend on x .

Now we repeat the above described procedure for this quadratic form:

$$\begin{aligned} f(x, y, z) &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \right)^2 + \frac{15}{8}y^2 - \frac{1}{8}z^2 + \frac{25}{4}yz = \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \right)^2 + \frac{15}{8} \left(y^2 + \frac{10}{3}yz \right) - \frac{1}{8}z^2 = \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \right)^2 + \frac{15}{8} \left(y^2 + \frac{10}{3}yz + \frac{25}{9}z^2 - \frac{25}{9}z^2 \right) - \frac{1}{8}z^2 = \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \right)^2 + \frac{15}{8} \left(y + \frac{5}{3}z \right)^2 - \frac{209}{72}z^2. \end{aligned}$$

Putting

$$\begin{cases} t_1 = \sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \\ t_2 = y + \frac{5}{3}z \\ t_3 = z \end{cases}$$

So, the canonical form is

$$g(t_1, t_2, t_3) = t_1^2 + \frac{15}{8}t_2^2 - \frac{209}{72}t_3^2$$

The rank of the quadratic form in the canonical form is equal the total number of square terms.

The rank of $f(x, y, z) =$ the rank of $g(t_1, t_2, t_3) = 3$.

Answer: 3