Answer on Question#40156 - Math – Linear Algebra:

Obtain an orthogonal basis with respect to the standard inner product for the subspace of $\mathbb{R}^4$ defined by $\{(x,y,z,w) \mid x+2y+z+3w=0,\ x-y-z=0\}$.

Solution.

Let $S=\{(x,y,z,w) \mid x+2y+z+3w=0,\ x-y-z=0\}$. Hence:

$$\begin{cases} x+2y+z+3w=0 \\ x-y-z=0 \end{cases} \Rightarrow \begin{cases} 3y+2z+3w=0 \\ x=y+z \end{cases} \Rightarrow \begin{cases} z=-\frac{3}{2}(y+w) \\ x=y+z \end{cases} \Rightarrow \begin{cases} z=-\frac{3}{2}(y+w) \\ x=-\frac{1}{2}(y+3w) \end{cases};$$

So:

$$S=\left\{\left(-\frac{y}{2}-\frac{3w}{2},y,-\frac{3y}{2}-\frac{3w}{2},w\right) \middle| \ y,w\in\mathbb{R}\right\}=\{yA+wB \mid y,w\in\mathbb{R}\},$$
$$\text{where } A=\left(-\frac{1}{2},1,-\frac{3}{2},0\right), B=\left(-\frac{3}{2},0,-\frac{3}{2},1\right);$$

We can conclude that:

$$S=V(A,B)\Rightarrow \dim(S)=2;$$

$\{A,B\}$ is a basis in S. Use Gram-Schmidt process to get an orthogonal basis $\{A',B'\}$:

$$A'=A=\left(-\frac{1}{2},1,-\frac{3}{2},0\right);$$

$$B'=B-\frac{(A,B)}{(A,A)}A=\left(-\frac{3}{2},0,-\frac{3}{2},1\right)-\frac{\frac{3}{4}+0+\frac{9}{4}+0}{\frac{1}{4}+1+\frac{9}{4}+0}\left(-\frac{1}{2},1,-\frac{3}{2},0\right)=$$

$$=\left(-\frac{3}{2},0,-\frac{3}{2},1\right)-\frac{6}{7}\left(-\frac{1}{2},1,-\frac{3}{2},0\right)=\left(-\frac{15}{14},-\frac{6}{7},-\frac{3}{14},1\right);$$

Hence, $\left\{\left(-\frac{1}{2},1,-\frac{3}{2},0\right),\left(-\frac{15}{14},-\frac{6}{7},-\frac{3}{14},1\right)\right\}$ is an orthogonal basis in S.