Answer on Question#40119 – Math – Linear Algebra:

Let $V = \mathbb{R}^5$. Find two subspaces U, W such that $U \oplus W = \mathbb{R}^5$.

Solution.

Let $U = \{(a, 0,0,0,0) \mid a \in \mathbb{R}\}, W = \{(0, b, c, d, e) \mid b, c, d, e \in \mathbb{R}\}$. Hence:

$$u_1 = (x, 0,0,0,0), u_2 = (y, 0,0,0,0) \in U, \alpha \in \mathbb{R} \Rightarrow u_1 + \alpha u_2 = (x + \alpha y, 0,0,0,0) \in U \Rightarrow$$

$$\Rightarrow U \text{ is a subspace;}$$

$$w_1 = (0, x, y, z, t), w_2 = (0, a, b, c, d) \in W, \alpha \in \mathbb{R} \Rightarrow$$

$$\Rightarrow w_1 + \alpha w_2 = (0, x + \alpha a, y + \alpha b, z + \alpha c, t + \alpha d) \in W \Rightarrow W \text{ is a subspace;}$$

$$\forall\, v = (x_1, x_2, x_3, x_4, x_5) \in V: v = u + w,$$

$$\text{where } u \in U, w \in W, u = (x_1, 0,0,0,0), w = (0, x_2, x_3, x_4, x_5);$$

$$u_1, u_2 \in U, w_1, w_2 \in W, u_1 + w_1 = u_2 + w_2 \Rightarrow u_1 - u_2 = w_2 - w_1 \Rightarrow$$

$$\Rightarrow (a_1, 0,0,0,0) - (a_2, 0,0,0,0) = (0, b_2, c_2, d_2, e_2) - (0, b_1, c_1, d_1, e_1) \Rightarrow$$

$$\Rightarrow (a_1 - a_2, 0,0,0,0) = (0, b_2 - b_1, c_2 - c_1, d_2 - d_1, e_2 - e_1) \Rightarrow a_1 - a_2 = 0 \Rightarrow u_1 = u_2 \Rightarrow$$

$$\Rightarrow w_1 = w_2;$$

Hence:

$$\forall\, v \in V\ \exists!\ u \in U, w \in W: v = u + w \implies V = U \oplus W.$$