Answer on question 40091 – Math – Real Analysis

For which real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^x}$$

converges?

Solution:

Consider the following series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}.$$

This series converges if $\alpha > 1$. The series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^x}$$

converges if

$$\lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^x} : \frac{1}{n^{\alpha}} < \infty, \quad \alpha > 1.$$

Thus we have

$$\lim_{n\to\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^x} : \frac{1}{n^{\alpha}} = \lim_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} \cdot n^{\alpha-x} =$$

$$= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n}\left(\sqrt{\frac{n+1}{n}}+1\right)} \cdot n^{\alpha-x} = \lim_{n\to\infty} \frac{1}{n^{0.5}\left(\sqrt{1+\frac{1}{n}}+1\right)} \cdot n^{\alpha-x} =$$

$$= \frac{1}{\sqrt{1+0}+1} \lim_{n\to\infty} n^{\alpha-x-0.5} = \frac{1}{2} \lim_{n\to\infty} n^{\alpha-x-0.5}.$$

We know that $\lim_{n\to\infty} n^{\alpha-x-0.5} < \infty$ if $(\alpha - x - 0.5) \le 0$. Thus we have next system of conditions

$$\begin{cases} \alpha - x - 0.5 \le 0, \\ \alpha > 1, \end{cases} \Rightarrow \begin{cases} \alpha - 0.5 \le x, \\ \alpha > 1, \end{cases} \Rightarrow x \ge \alpha - 0.5 > 1 - 0.5 = 0.5$$

Thus we have that

$$\boxed{x > 0.5}$$

Answer:

$$\boxed{x > 0.5}$$