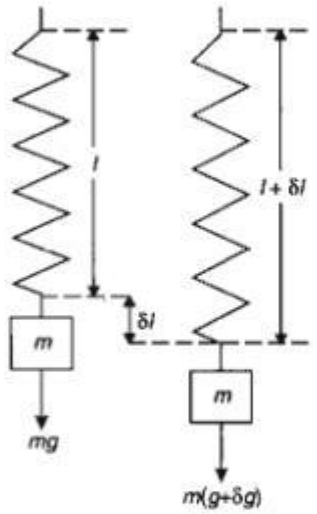
Answer on Question #40031, Math, Differential Calculus

A mass weighing 39.5 kg. stretches a spring 1/4m. At t=0 , the mass is released from a point 3/4m below the equilibrium position with an upward velocity of (5/4)m/sec. Determine the function x(t) that describes the subsequent free motion.



Solution.

$$m = 39.5\ kg$$

$$L = \frac{1}{4}m$$

$$x_0 = \frac{3}{4}m$$

$$v_0 = x_0' = \frac{5}{4}\frac{m}{s}$$

$$g = 9.8\frac{m}{s^2}$$

For a mass m suspended from a spring, the weight is jointly proportional to the distance stretched and a constant: $F = ks$ (Hooke's Law) or weight $= kx$.

If the motion of the mass is free and undamped (also called simple harmonic), it is described by:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Circular frequency: $\omega = \sqrt{\frac{k}{m}}$

To find the equation of motion:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Let's $x = e^{\lambda t}$

Then

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0$$

$$\lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

$$x = C_1 e^{-i\omega t} + C_2 e^{i\omega t} = A \sin \omega t + B \cos \omega t$$

$$x(0) = x_0 = \frac{3}{4} - initial\ displacement:$$

$$positive\ if\ BELOW\ equlibrum\ point$$

$$negative\ if\ ABOVE\ equlibrum\ point$$

$$x'(0) = x_0' = \frac{5}{4} - initial\ velocity:$$

$$positive\ DOWNWARD$$

$$negative\ UPWARD$$

$$= 0\ if\ from\ rest$$

So find $A\ and\ B$:

$$x(0) = A \sin \omega \cdot 0 + B \cos \omega \cdot 0 = B = \frac{3}{4}$$

$$x'(t) = A\omega \cos \omega t - B\omega \sin \omega t$$

$$x'(0) = A\omega \cos \omega \cdot 0 - B\omega \sin \omega \cdot 0 = A\omega = \frac{5}{4}$$

So

$$A = \frac{5}{4\omega}$$

$$B = \frac{3}{4}$$

Find ω:

$$\omega = \sqrt{\frac{k}{m}}$$

From Hooke's Law:

$$F = kL = mg$$

where $L - point\ of\ equilibrium\ , L = \frac{1}{4}$

$$k \cdot \frac{1}{4} m = 39.5\ kg \cdot 9.8\ \frac{m}{s^2}$$

$$k = 39.5 \cdot 9.8 \cdot 4 = 1548.4 \frac{N}{m} - \ spring\ stiffness$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}} = 2\sqrt{g} \approx 6.3 \frac{rad}{s}$$

Therefore,

$$x = \frac{5}{4\omega} \sin \omega t + \frac{3}{4} \cos \omega t = 0.2 \sin 6.3t + 0.75 \cos 6.3t$$

Answer:

$$x(t) = 0.2 \sin 6.3t + 0.75 \cos 6.3t$$