Answer on Question #40029, Math, Differential Calculus

Question:

A series RLC circuit with R-6 ohm, C-0.02 Farad and L=0.1 has no applied voltage. Find the subsequent current in the circuit if the initial charge, on the capacitor is q and the initial current is zero.

Answer:

Kirchhoff's voltage law:

$$u_R + u_L + u_C = 0$$

where $u_R,\ u_L,\ u_C$ are the voltages across R, L and C respectively.

Substituting in the constitutive equations:

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0$$

Differentiating and dividing by L:

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

This can usefully be expressed in a more generally applicable form:

$$\frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + {\omega_0}^2 i(t) = 0$$

$$\alpha = \frac{R}{2L}, \omega_0 = \frac{1}{\sqrt{LC}}$$

The differential equation has the characteristic equation:

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

The roots of the equation in s are:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$$

The general solution of the differential equation is an exponential in either root or a linear superposition of both

$$i(t) = Ae^{s_1 t} + Be^{s_2 t}$$

The initial current is zero:

$$i(0) = Ae^0 + Be^0 = 0$$

Therefore: $A = -B$

$$i(t) = A(e^{s_1 t} - e^{s_2 t}) = Ae^{-\alpha t}\left(e^{\sqrt{\alpha^2-{\omega_0}^2}t} - e^{-\sqrt{\alpha^2-{\omega_0}^2}t}\right)$$

$$\sqrt{\alpha^2 - {\omega_0}^2} = \sqrt{-100}, \alpha = 30$$

therefore $\left(e^{10it} - e^{-10it}\right) = 2i\sin 10t,\, 2iA = A'$:

$$i(t) = A'e^{-30t}\sin 10t$$

The initial charge on the capacitor is q and initial current is zero:

$$L\frac{di(t)}{dt}|_{t=0} + \frac{q}{C} = 0$$

$$\frac{di(t)}{dt}|_{t=0} = -\frac{q}{0.001} = 10A'$$

$$A' = 0.01q$$

Therefore:

$$i(t) = 0.01qe^{-30t}\sin 10t$$