Answer on Question #40025, Math, Linear Algebra

Investigate the nature of the conic $5x^2 + 4xy + 5y^2 + 2x + y = 1$.

Solution.

If (α, β) is the centre of the conic,

$$10\alpha + 4\beta + 2 = 0$$

$$4\alpha + 10\beta + 1 = 0$$

So,

$$\alpha = -\frac{4}{21}$$

$$\beta = -\frac{1}{42}$$

Putting $x = X - \frac{4}{21}$, $y = Y - \frac{1}{42}$ in equation, transferring the origin to $(-\frac{4}{21}, -\frac{1}{42})$, the equation of the conic becomes

$$5(X - \frac{4}{21})^2 + 4(X - \frac{4}{21})(Y - \frac{1}{42}) + 5\left(Y - \frac{1}{42}\right)^2 + 2(X - \frac{4}{21}) + Y - \frac{1}{42} = 1$$

$$5X^2 + 4XY + 5Y^2 = \frac{101}{84}$$

Changing the axes to lines though the origin, making an angle f with the old axes, i.e. substituting

$$X = X' \cos f - Y' \sin f$$

$$Y = X' \sin f + Y' \cos f$$

The equation becomes:

$$5(X' \cos f - Y' \sin f)^2 + 4(X' \cos f - Y' \sin f)(X' \sin f + Y' \cos f) + 5(X' \sin f + Y' \cos f)^2 = \frac{101}{84}$$

If f is so chosen that the coefficient of $X'Y'$ is zero, i.e.

$$-10 \cos f \sin f + 4 \cos^2 f - 4 \sin^2 f + 10 \sin f \cos f = 0$$

$$4 \cos 2f = 0$$

$$\cos 2f = 0$$

$$f = \frac{\pi}{4}$$

And now the equation becomes

$$\frac{5}{2}(X' - Y')^2 + 2(X'^2 - Y'^2) + \frac{5}{2}(X' + Y')^2 = \frac{101}{84}$$

or

$$7X'^2 + 3Y'^2 = \frac{101}{84}$$

Answer:

Hence, the conic represented by the equation is an ellipse.