Answer on Question #40020, Math, Linear Algebra

Find the rank of the quadratic form 2x2+2y2+3z2-xy+4yz+5xz.

Solution

We will find a canonical expression of the quadratic form $f(x, y, z) = 2x^2 + 2y^2 + 3z^2 - xy + 4yz + 5xz$.

We will find a form $g(t_1, t_2, t_3)$ by a procedure known as Lagrange's Reduction which consists essentially of repeated completing of the square. First of all, we collect all the terms with x and complete the resulting expression to a square.

$f(x, y, z) = (2x^2 - xy + 5xz) + 2y^2 + 3z^2 + 4yz = \left(2x^2 + \frac{1}{8}y^2 + \frac{25}{8}z^2 - xy + 5xz - \frac{5}{4}yz - \frac{1}{8}y^2 - \frac{25}{8}z^2 + \frac{5}{4}yz\right) + 2y^2 + 3z^2 + 4yz = \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z\right)^2 + \frac{15}{8}y^2 - \frac{1}{8}z^2 + \frac{25}{4}yz.$

The quadratic form $(\frac{15}{8}y^2 - \frac{1}{8}z^2 + \frac{25}{4}yz)$ is in two variables y and z and does not depend on x. Now we repeat the above described procedure for this quadratic form:

$$\begin{aligned} f(x, y, z) &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z\right)^2 + \frac{15}{8}y^2 - \frac{1}{8}z^2 + \frac{25}{4}yz \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z\right)^2 + \frac{15}{8}\left(y^2 + \frac{10}{3}yz\right) - \frac{1}{8}z^2 \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z\right)^2 + \frac{15}{8}\left(y^2 + \frac{10}{3}yz + \frac{25}{9}z^2 - \frac{25}{9}z^2\right) - \frac{1}{8}z^2 \\ &= \left(\sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z\right)^2 + \frac{15}{8}\left(y + \frac{5}{3}z\right)^2 - \frac{209}{72}z^2. \end{aligned}$$

Putting

$$\begin{cases} t_1 = \sqrt{2}x - \frac{1}{2\sqrt{2}}y + \frac{5}{2\sqrt{2}}z \\ t_2 = y + \frac{5}{3}z \\ t_3 = z \end{cases}$$

we transform the quadratic form $f(x, y, z)$ into its canonical expression

$$g(t_1, t_2, t_3) = t_1^2 + \frac{15}{8}t_2^2 - \frac{209}{72}t_3^2.$$

The rank of the quadratic form in the canonical form is equal the total number of square terms.

The rank of $f(x, y, z)$ = the rank of $g(t_1, t_2, t_3)$ = 3.

Answer: 3.